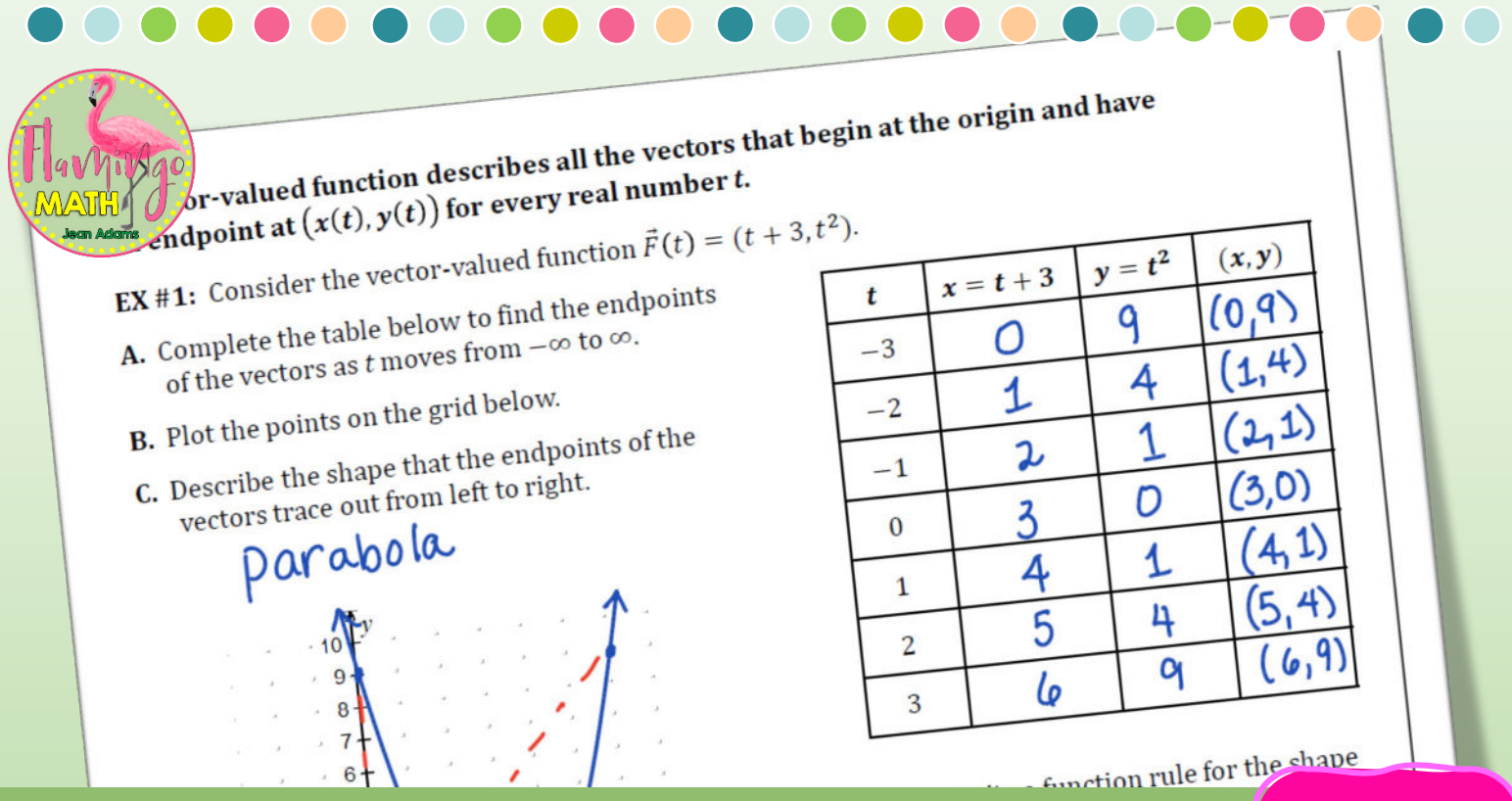
AP® Precalculus

Vector-Valued Functions



t	$x = t+3$	$y = t^2$	(x, y)
−3	0	9	(0,9)
−2	1	4	(1,4)
−1	2	1	(2,1)
0	3	0	(3,0)
1	4	1	(4,1)
2	5	4	(5,4)
3	6	9	(6,9)

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Lesson 9: Vector-Valued Functions

In this lesson we will explore the vector form of the parametric equations $x = f(t)$ and $y = g(t)$, where f and g are continuous functions of t on an interval I. At this point, we have only considered functions where the domain and range are sets of real numbers, or ***real-valued functions***.

A ***vector-valued function*** receives a scalar as an input and returns a vector as an output. We will work with vector-valued functions on a two-dimensional plane in this course, but they are also valuable tools in three-dimensional space for solving many problems from physics.

Recall that plane curves were defined by coordinates at time t as $(f(t), g(t))$. Now we want to represent motion along a curve by giving a definition to a **vector-valued function** where the domain consists of reals numbers and maps the range to a set of vectors.

Definition of a Vector-Valued Function

In two-dimensions, the position of a particle moving in a plane that is given by the parametric function $f(t) = (x(t), y(t))$ may be represented as a **vector-valued function** in the form

$$\vec{F}(t) = \langle x(t), y(t) \rangle \text{ or } \vec{F}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

Where the component functions f and g are real-valued functions of the parameter t.

*The **magnitude of the position vector at time t*** gives the distance of the particle from the origin.

A vector-valued function describes all the vectors that begin at the origin and have an endpoint at $(x(t), y(t))$ for every real number t.

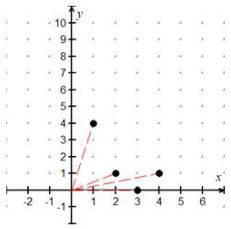
EX #1: Consider the vector-valued function $\vec{F}(t) = (t + 3, t^2)$.

A. Complete the table below to find the endpoints of the vectors as t moves from $-\infty$ to ∞.

B. Plot the points on the grid below.

C. Describe the shape that the endpoints of the vectors trace out from left to right.

t	$x = t + 3$	$y = t^2$	(x, y)
-3			
-2			
-1			
0			
1			
2			
3			

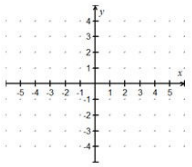


D. Can you write a function rule for the shape that is traced out by the vector endpoints?

Comparing Functions

In the first example, our function traced the shape of a parabola. With the function rule, $y = (x - 3)^2$ each point along the curve simply represents a point in space. However, in the vector-valued function: $\vec{F}(t) = (t + 3, t^2)$, each point represents a vector, so we can use the information on the graph to find magnitudes and angles of direction or perform operations on vectors.

EX #2: Express the parametric equations $x = 5\cos t$, $y = 3\sin(t)$ as a vector-valued function. Sketch the curve.



Finding Domain of Vector-Valued Functions

The domain of a vector function is the intersection of the domains of the component functions, f and g. To find the domain for a vector-valued function $\vec{F}(t) = \langle x(t), y(t) \rangle$ we must find the values of t for which $\vec{F}(t)$ is continuous at both $f(t)$ and $g(t)$.

EX #3: Given the vector equation $\vec{F}(t) = (t^2 + 1)\mathbf{i} + ln(2 - t)\mathbf{j}$, find the following:

A. $x = f(t)$ and $y = g(t)$

B. The domain of F

Velocity of a Particle

The vector-valued function $v(t) = \langle x(t), y(t) \rangle$ can be used to express the velocity of a particle moving in a plane at different times, t. At time t, the sign of $x(t)$ indicates if the particle is moving right or left. The sign of $y(t)$ indicates if the particle is moving up or down. The magnitude of the velocity vector at time t gives the speed of the particle.

Particle Motion and Speed

EX #4: The velocity of a particle moving along a path can be modeled by the vector-valued function $\mathbf{v}(t) = \langle 2t^2, t + 3 \rangle$.

A. Find the $\mathbf{v}(1)$ and describe the motion of the particle.

B. Find the speed of the particle at this time.

EX #5: Find the magnitude and angle of $\vec{G}(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}$ for $t = 4$.

Displacement

Let $\vec{F}(t)$ be a vector-valued function and let $t_0 < t_1$ be values in the domain.

The *displacement* $\vec{d}$ of $\vec{F}$ from $t = t_0$ to $t = t_1$ is $\vec{d} = \vec{F}(t_1) - \vec{F}(t_0)$

C. Sketch the displacement vector.

D. Explain the meaning in relation to the object. .

Average Rate of Change

Knowing distance from the starting point allows you to compute the **average rate of change.** Let $\vec{F}(t)$ be a vector-valued function where each of its component functions is continuous on its domain and let $t_0 < t_1$.

The *average rate of change* of $\vec{F}(t)$ on $[t_0, t_1]$ is

$$\textbf{AROC} = \frac{\vec{F}(t_1) - \vec{F}(t_0)}{t_1 - t_0}$$

EX #7: An object moves along the elliptical pathway modeled by $\vec{F}(t) = \langle 4\cos(t), 5\sin(t) \rangle$ on $0 \le t \le 2\pi$.

A. Find the average rate of change of $\vec{F}(t)$ on $[1, 2]$

B. Explain the meaning in relation to the object. What is the motion of the object during this interval of time?

C. Find the distance of the object from the origin at $t = 2$,

Guided Notes

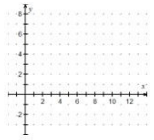
4.9 **Vector-Valued Functions Practice Set 1**

Name _____

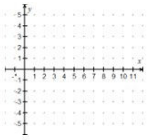
Date _____ Period _____

Problems 1-4, Consider the vector-valued functions for $-2\pi \le t \le 2\pi$. Sketch the graph that the endpoints of the vectors trace out from left to right. State the domain.

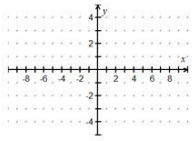
1. $\vec{G}(t) = \langle 2t^2, t+4 \rangle$



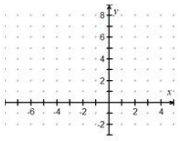
2. $\vec{H}(t) = \left\langle t^2, \frac{1}{t} \right\rangle$



3. $\vec{F}(t) = \langle 5\sin t, 2\cos t \rangle$



4. $\vec{K}(t) = \langle t-2, t^2 \rangle$



Problems 5-8, Find the magnitude and angle for each function at the value for time, t.

5. $\vec{G}(t) = \langle 2t^2, t+4 \rangle$ at $t = 2$

6. $\vec{H}(t) = \left\langle t^2, \frac{1}{t} \right\rangle$ at $t = 5$

7. $\vec{F}(t) = \langle 5\sin t, 2\cos t \rangle$ at $t = 0$

8. $\vec{K}(t) = \langle t-2, t^2 \rangle$ at $t = 4$

Problems 9 – 12, Write the component functions and find the domain of each vector-valued function.

9. $\boldsymbol{F}(t) = \frac{1}{t+2}\mathbf{i} + \frac{2}{3}t\,\mathbf{j}$

10. $\boldsymbol{G}(t) = \sqrt{t-2}\,\mathbf{i} + \ln t\,\mathbf{j}$

11. $\boldsymbol{H}(t) = \langle t^2, \sqrt{9-t^2} \rangle$

12. $\boldsymbol{K}(t) = \langle e^t, \ln(t+3) \rangle$

Problems 13 – 14, Evaluate the vector-valued function at each value of t, if possible.

13. $\boldsymbol{F}(t) = \langle \cos t, 2\sin t \rangle$

A. $\boldsymbol{F}\left(\frac{\pi}{3}\right)$

B. $\boldsymbol{F}(\pi)$

C. $\boldsymbol{F}(0)$

14. $\boldsymbol{G}(t) = \langle \sqrt{t+1}, t^{3/2} \rangle$

A. $\boldsymbol{G}(0)$

B. $\boldsymbol{G}(-2)$

C. $\boldsymbol{G}(k-1)$

Problems 15-18, Given the velocity of a particle moving along a path, find the speed of the particle and describe the motion of the particle at the given time.

15. $\mathbf{v}(t) = \langle t^{3/2}, -4\sqrt{t} \rangle$ at $t = 4$

16. $\mathbf{v}(t) = \langle t^2, \ln(t-1) \rangle$ at $t = 3$

17. $\mathbf{v}(t) = (t^2-2)\mathbf{i} - 3t\mathbf{j}$ at $t = 2$

18. $\mathbf{v}(t) = 5\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j}$ at $t = 1$

Problems 19-20, find the displacement and the average rate of change on the given interval.

19. $\vec{F}(t) = (t+3)\mathbf{i} + t^2\mathbf{j}$ for $[0,2]$

20. $\vec{G}(t) = \left(-\frac{1}{2}t^2\right)\mathbf{i} + (t-1)\mathbf{j}$ for $[1,2]$

Practice Sets

4.9 Vector-Valued Functions **Form A**

Name _____

Date _____Period _____

Use the vector-valued function for problems 1 and 2.

$G(t) = \langle t - 5, 3t^2 \rangle$

1. Find the magnitude at $t = 4$

2. Find the angle at $t = 4$.

Use the vector-valued function for problems 3 and 4.

$F(t) = (-2t^2)\mathbf{i} + (t + 3)\mathbf{j}$

3. Find the displacement over the interval $[2, 5]$

4. Find the average rate of change over the interval $[2, 5]$

Grade:

4.9 Vector-Valued Functions **Form B**

Name _____

Date _____Period _____

Use the vector-valued function for problems 1 and 2.

$G(t) = \langle t - 3, 2t^2 \rangle$

1. Find the magnitude at $t = 5$

2. Find the angle at $t = 5$.

Use the vector-valued function for problems 3 and 4.

$F(t) = (-3t^2)\mathbf{i} + (t + 4)\mathbf{j}$

3. Find the displacement over the interval $[1, 5]$

4. Find the average rate of change over the interval $[1, 5]$

Grade:

Daily Lesson Quiz