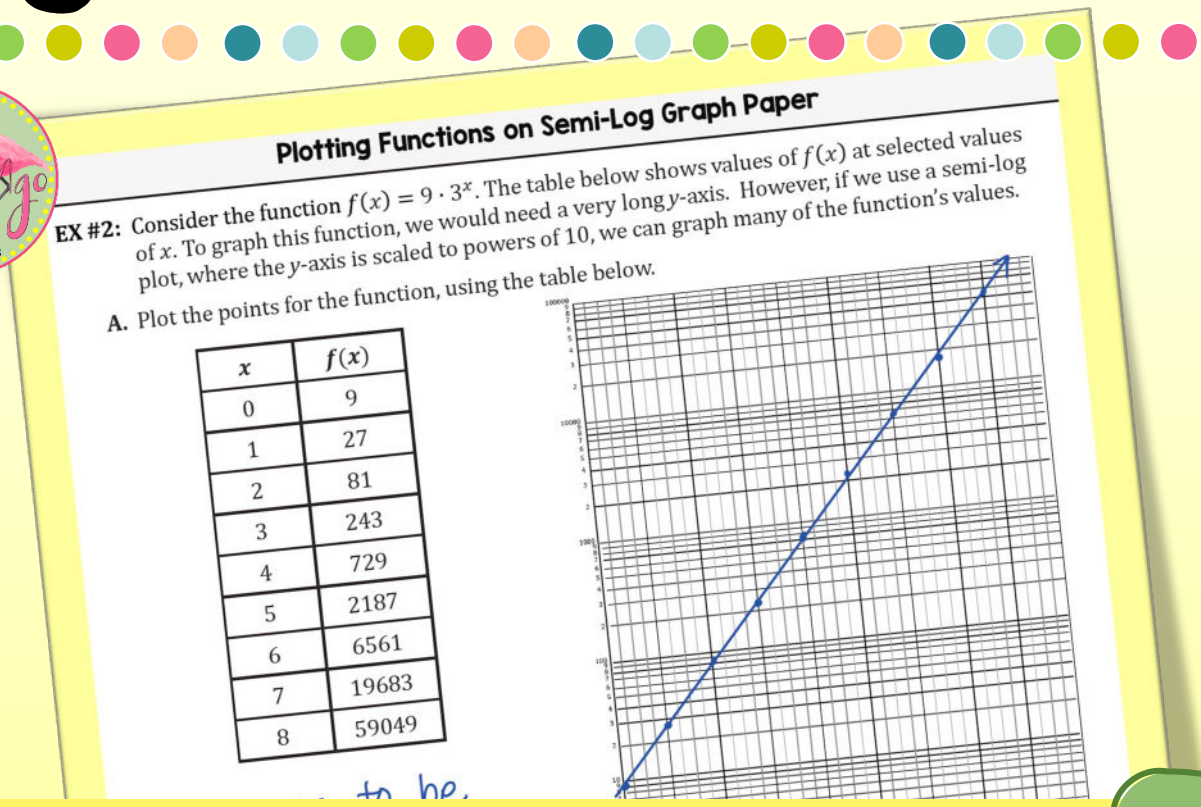
AP® Precalculus

Exponential and Logarithmic Functions



Plotting Functions on Semi-Log Graph Paper

EX #2: Consider the function $f(x) = 9 \cdot 3^x$. The table below shows values of $f(x)$ at selected values of x. To graph this function, we would need a very long y-axis. However, if we use a semi-log plot, where the y-axis is scaled to powers of 10, we can graph many of the function's values.

A. Plot the points for the function, using the table below.

x	$f(x)$
0	9
1	27
2	81
3	243
4	729
5	2187
6	6561
7	19683
8	59049

Guided Notes 2

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Pre-AP PRECALCULUS CURRICULUM

Scope and Sequence

Unit 2: Exponential and Logarithmic Functions

Lesson	Skills & Learning Objectives
[2.1] Change in Arithmetic and Geometric Sequences	▪ Recognize, write and find the *n*th terms of arithmetic sequences ▪ Use arithmetic sequences to model real-world problems ▪ Recognize, write and find the *n*th terms of geometric sequences ▪ Use geometric sequences to model real-world problems
[2.2] Change in Linear and Exponential Functions	▪ Construct functions of the real numbers that are comparable to arithmetic and geometric sequences ▪ Describe similarities and differences between linear and exponential functions ▪ Solve real-world applications using arithmetic and geometric sequences
[2.3] Exponential Functions	▪ Investigate the basic behaviors of exponential functions and their graphs ▪ Analyze exponential graphs for key characteristics such as domain, range, end-behavior, continuity, symmetry, asymptotes, increase and decrease.
[2.4] Exponential Function Manipulation	▪ Use properties of exponents and logarithms to simplify and evaluate expressions ▪ Apply the properties of logarithms to evaluate expressions and graph functions ▪ Understand the characteristics and effects of constants in the exponential model $y = ab^{kx}$

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Scope and Sequence

Unit 2: Exponential and Logarithmic Functions

Lesson	Skills & Learning Objectives
[2.5] Exponential Function Context and Data Modeling	▪ Construct a model for scenarios that involve proportional output values related to real-world scenarios ▪ Apply exponential models about data sets in contextual scenarios ▪ Construct exponential function models using regression equations and technology
[2.6] Competing Function Model Validation	▪ Find an appropriate regression model for a data set. ▪ Use contextual clues to determine an appropriate model. ▪ Recognize a correct regression model using the graph of the residuals. ▪ Find the residual errors of a model and analyze the findings as underestimating or overestimating the data set in context.
[2.7] Composition of Functions	▪ Perform operations on functions ▪ Find a composition of one function with another function ▪ Rewrite a given function as a composite of two or more functions ▪ Recognize that the composition of functions is not commutative ▪ Recognize the composite function $f(g(x))$ uniquely maps input values of g to output values of f , dependent on the domain restrictions of f and g.
[2.8] Inverse Functions	▪ Verify two functions are inverses algebraically and graphically ▪ Use the Horizontal Line Test to determine if functions are one-to-one ▪ Find inverse functions informally and algebraically ▪ Use function notation to describe real-world scenarios ▪ Analyze and choose the appropriate method to find the inverse of a relation or function

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Scope and Sequence

Unit 2: Exponential and Logarithmic Functions

Lesson	Skills & Learning Objectives
[2.9] Logarithmic Expressions	▪ Apply the properties of logarithms to evaluate expressions with and without technology ▪ Use properties of exponents and logarithms to simplify and evaluate expressions ▪ Understand each unit of the logarithmic scale is a multiplicative change of the base $10^0, 10^1, 10^2, \ldots$
[2.10] Inverses of Exponential Functions	▪ Use the general form of a logarithmic function to evaluate and graph. ▪ $f(x) = \log_b x$ and $g(x) = b^x$, if $g\big(f(x)\big) = f\big(g(x)\big) = x$ then $f(x)$ and $g(x)$ are inverse functions. ▪ Graphs of inverse functions are reflections across the line $y = x$. ▪ Analyze exponential and logarithmic graphs for domain, range, end-behavior, concavity, asymptotes, increase and decrease. ▪ Determine whether a function behavior is exponential or logarithmic from proportionality of input and output values.
[2.11] Logarithmic Functions	▪ Understand the domain of a logarithmic function in general form is any real number greater than zero, and the range is all real numbers. ▪ Identify key characteristics of logarithmic functions and transformations. ▪ Determine whether logarithmic functions are increasing decreasing, concave up, or concave down. ▪ Logarithmic functions do not have extrema, except on a closed interval, and their graphs do not have inflection points. ▪ Analyze and use limit notation for end behavior.

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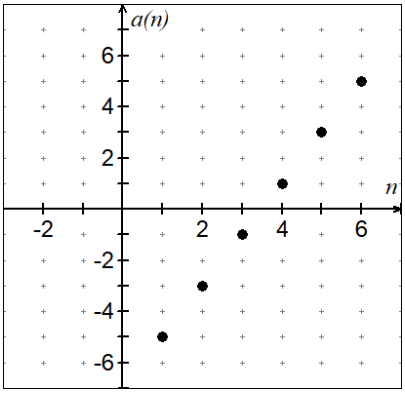
Lesson 2-A: Change in Linear and Exponential Functions

Let's look at the relationships between linear functions and arithmetic sequences. We know that a linear function of the form $y = mx + b$ describes a continuous function whose domain and range is all real numbers. The y-intercept is the point $(0, b)$ and the rate of change of the function is the value of m which represents the slope of the line over its domain. Arithmetic sequences can be represented by linear functions. However, sequences and their corresponding functions may have different domains.

Generating a Linear Function from an Arithmetic Sequence

EX #1: Use the graph of the sequence shown at right to



A. Complete the table below.

n	0	1	2	3	4	5	6
a_n		-5					

B. Find the difference d and write an explicit rule, a_n.

C. Use the information from part (B) to write a linear function that corresponds to the nth term expression.

EX #2: Given the arithmetic sequence where $a_1 = 17$ and $a_8 = -39$, state the domain and range of the sequence.

n	1	2	3	4	5	6	7	8
a_n	17							-39

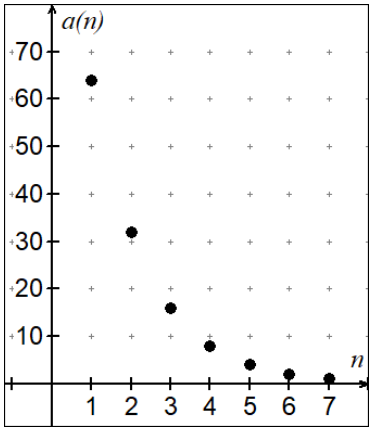
EX #3: Write a linear function whose domain is all real numbers that will contain the domain of the sequence in EX #2. Show your steps for writing the function.

Lesson 2-B: Change in Linear and Exponential Functions

We have seen that when the output values of a function change at a constant rate, the function is linear. Now let's look at what happens when the output values change proportionally. When your sequence is geometric, the rate of change will increase (or decreases) as the value of *n* increases.

Generating an Exponential Function from a Geometric Sequence

EX #8: Use the graph of the sequence shown at right to



A. Complete the table below.

n	1	2	3	4	5	6	7
a_n	64	32			4		

B. Find the common ratio *r* and write an explicit rule, a_n.

C. Use the information from part (B) to write an exponential function that corresponds to the *n*th term expression.

EX #9: Given the geometric sequence where $a_1 = 144$ and $a_3 = 36$, state the domain and range of the sequence.

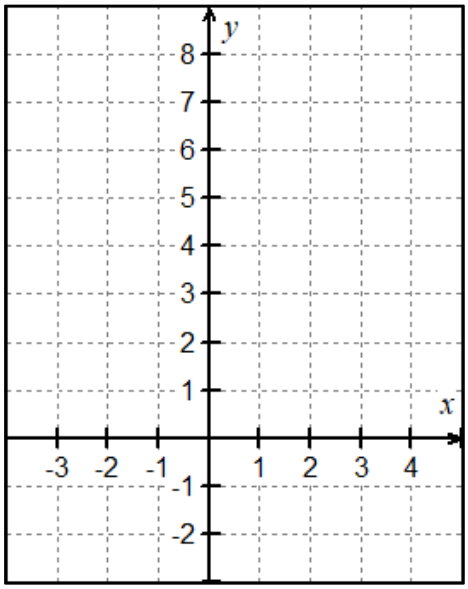
n	1	2	3	4	5	6	7
a_n	144		36			$\frac{9}{2}$	

EX #10: Write an exponential function whose domain is all real numbers that will contain the domain of the sequence in EX #9. Show your steps for writing the function.

Lesson 4-B: Exponential Function Manipulation

EX #5: Graph $f(x) = 2^{-x}$ on the grid at right. Then, use the graph to obtain the graph of $g(x) = 2^{-2x}$

x	$f(x) = 2^{-x}$	$g(x) = 2^{-2x}$
-2		
-1		
0		
1		
2		

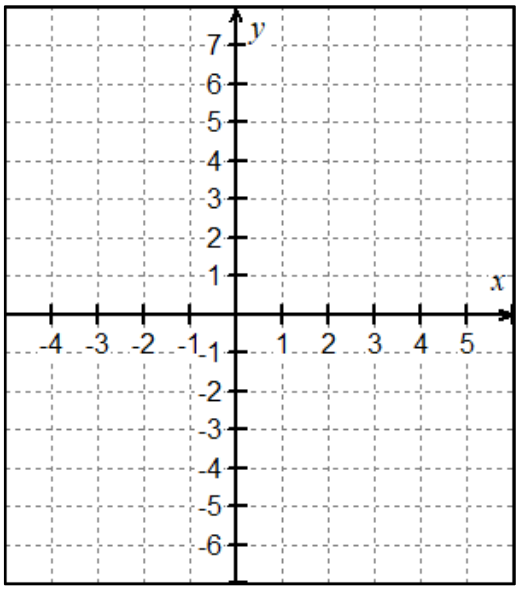


EX #6: Show that the function $g(x) = 2^{-2x}$ can be written in the form of $h(x) = \left(b^k\right)^x$ using the transformation properties in the table shown on the previous page. Describe the transformation of the y-intercept from the parent function $f(x) = 2^{-x}$ for $g(x)$ and $h(x)$.

EX #7: Use the grid at right to graph $f(x) = -2^x$ and $g(x) = -2^x + 3$.

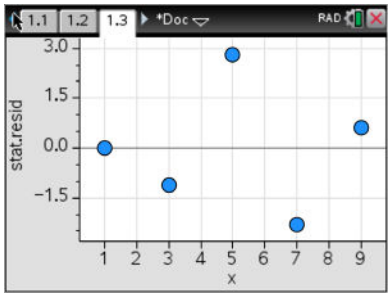
A. Label the y-intercept for each function.

B. Describe the transformations for each function compared to the parent function $y = 2^x$

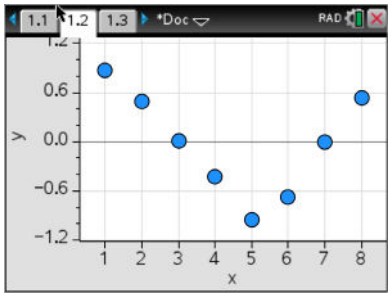


Residual Plots

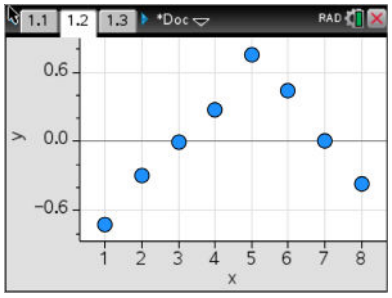
A **residual plot** is a scatter plot that shows the residuals on the vertical axis and the independent variable on the horizontal axis. We will use residual plots to help us decide whether the model chosen is an appropriate model. Notice the residual plot (below) for our example where $\hat{y} = 2.05x + 4.95$ that the residuals are randomly placed above and below the x-axis.
You will see either random residual plots, appropriate for linear regressions, or patterned residual plots for non-linear regressions.



Random
Linear Model



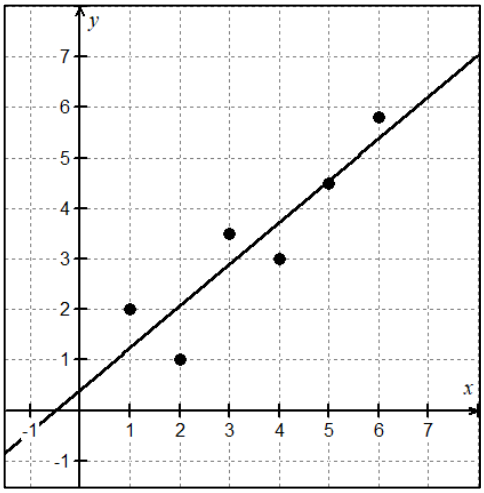
U-Shaped Pattern
Non-Linear Model



Inverted U-Shaped Pattern
Non-Linear Model

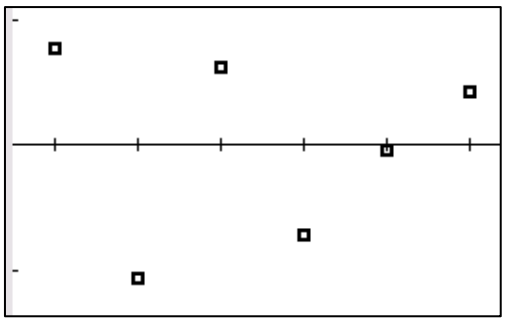
How to Interpret a Residual Plot

EX #3: Let's look at a data plot with the least squares regression line and analyze the associated residual plot.



A. Use a red pencil to draw line segments from the data points to the line of best fit in the graph at right. $\hat{y} = 0.83x + 0.4$ These ***residual line segments*** fall above and below the regression line. What does this tell you about the values of residuals?

B. How do we know if our predicted line overestimates or underestimates our data? Use the residual plot of the data set to complete the statements.



The predicted line will _____
data when a residual value is positive.

The predicted line will _____
Data when a residual value is negative.

EX #8: While driving down a dirt road you punctured the tire of your truck. The table below shows the pressure, y, measured in pounds per square inch (psi) if air in the tire at any given time, x, measured in seconds after the tire was punctured.

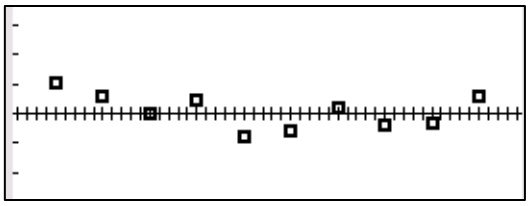
x (sec)	5	10	15	20	25	30	35	40	45	50
y (psi)	28	22	17	14	10	8	7	5	4	4

A. Find an exponential function in the form of $y = ab^x$ that best fits the data.

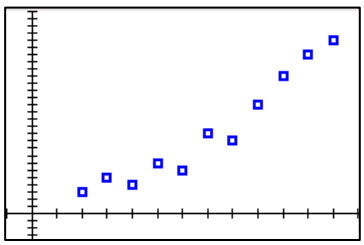
B. Plot the equation and scatter plot on the same screen. Does the exponential function fit the data points? Explain.

C. Using the model, what was the pressure of the tire before it was punctured?

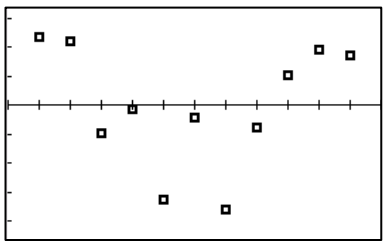
D. The residual graph is shown at right. Does this match your calculation? What does it tell you about the validity of your regression function?



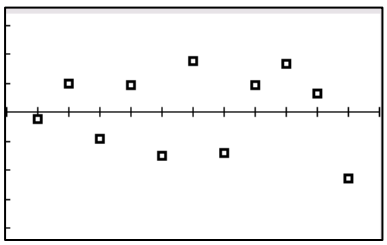
EX #9: A set of data has an upward trend, shown in the plot below. It is not clear whether the best fitting function is linear or exponential. Use the residual plots to determine which regression function would best model the data set. Explain why.



Data Plot



Linear Function Residuals



Exponential Function Residuals

Lesson I5-A: Semi-Log Plots

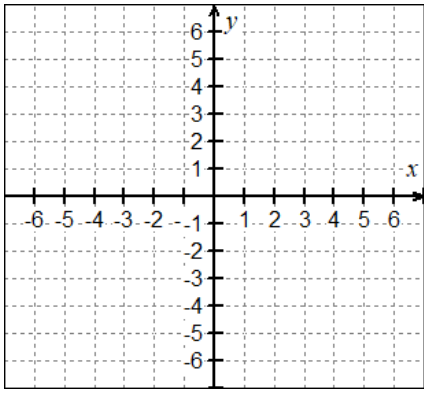
Semi-log plots are a useful tool for visualizing data that covers a wide range of values. A ***semi-log plot*** is a graph that combines a logarithmic scale for one axis with a linear scale for the other axis. It is used to plot data that covers a wide range of values, where a linear plot would make it difficult to see the details. The semi-log plot will create a graph that reveals the details of the data while still showing the overall trend. In this lesson, we will learn how to create and interpret semi-log plots.

Plotting Functions on Semi-Log Graph Paper

EX #1: Let's consider graphing the function $f(x) = x$ on two different types of graph paper.

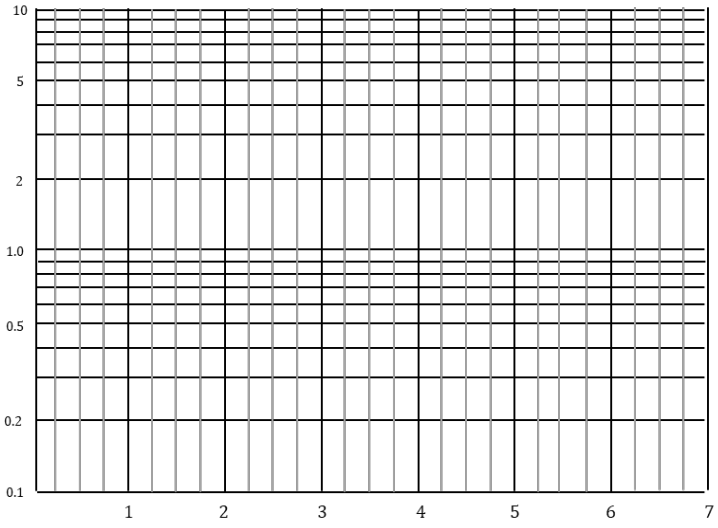
A. On a linear axes, graph $y = x$ for table below.

x	-2	-1	0	1	2	3	4	5	6
$f(x)$	-2	-1	0	1	2	3	4	5	6



B. Use the semi-logarithmic axes, at right, to plot the graph of $y = x$.

Notice that the vertical axis is a logarithmic scale (base 10) so there are no negative values for y since we can't find the log of a negative number. The horizontal axis is a linear scale.



C. What do you notice about the graph of the linear function on this semi-log plot?

Mathematically, we treat a logarithmic axis as a transformation of the variables. For our course, we will plot data pairs (x, y) in a semi-log plot where we are working with the transformed data pairs $(x, \log y)$ or $(x, \ln y)$. Think about the log y as a variable. The logarithmic scale is a great tool to compress a large range of data into a small space and be able to detect small variations in the values of the data points in a set.

Lesson I5-B: Semi-Log Plots

In the first lesson on semi-log plots, we knew the base of the exponential function. But, if we have real world data and do not know the function or the base, we will take the logarithm of the output values and then we can graph using a logarithmically scaled vertical axis. So, whether we have the exponential function of raw data, in either case, we can linearize the data by transforming the *y*-values to a logarithmic scale.

Using Raw Data

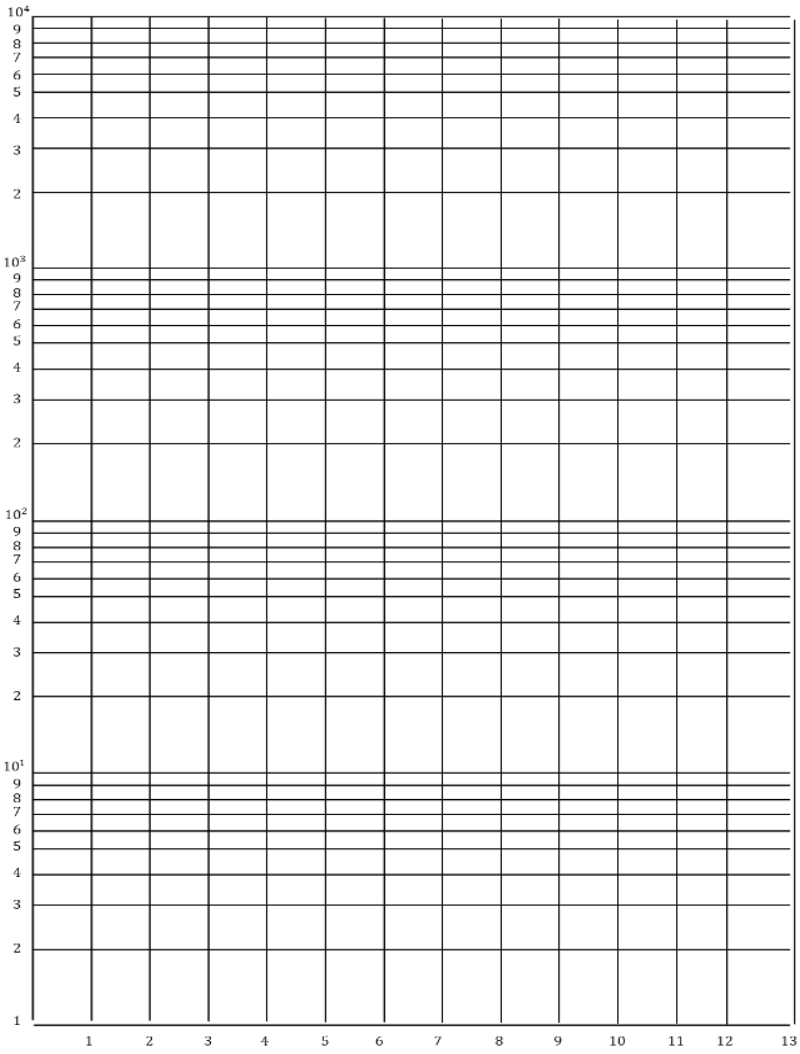
EX #7: The table below gives values for a function $f(x)$ at selected values of t.

x	0	2	4	5	6	7	8	10	12	13
$f(x)$	0.3	1.2	5.8	12.6	28.8	65.4	156.2	820.3	4352.6	9942.5

A. Graph the data on the semi-log plot below. Label your axes.

B. What does this graph say about the data?

C. Approximate the values of $f(3)$ and $f(9)$ using the graph.



Digging Deeper into Semi-log Plots

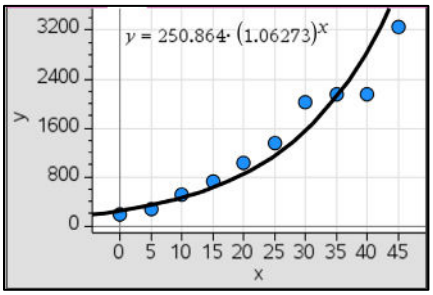
EX #9: The table at right shows the amount of revenue R (in billions of dollars) collected by the IRS for selected years from 1970 to 2015.

Year	Revenue, R
1970	192.8
1975	279.1
1980	517.1
1985	734.0
1990	1030.0
1995	1350.0
2000	2030.0
2005	2150.0
2010	2160.0
2015	3250.0

A. Consider how the revenue by year seems to change for each 5-year period. Is the change by a constant difference or a constant ratio? Explain.

B. Find an exponential regression model for the data. Graph is shown below.

C. Interpret the meaning of 250.864 in the regression equation.



D. Interpret the meaning of 1.06273 in the regression equation.

E. Use the exponential model to predict the Tax Revenue for the year 2025.

F. The data modeled by an exponential function can be linearized by finding $(x, \ln y)$. Complete the table below.

Years since 1970	0	5	10	15	20	25	30	35	40	45
$\ln(y)$ Revenue										

G. Graph the linearized data from the table above and find a linear regression equation, where $\hat{y} = \ln y$.