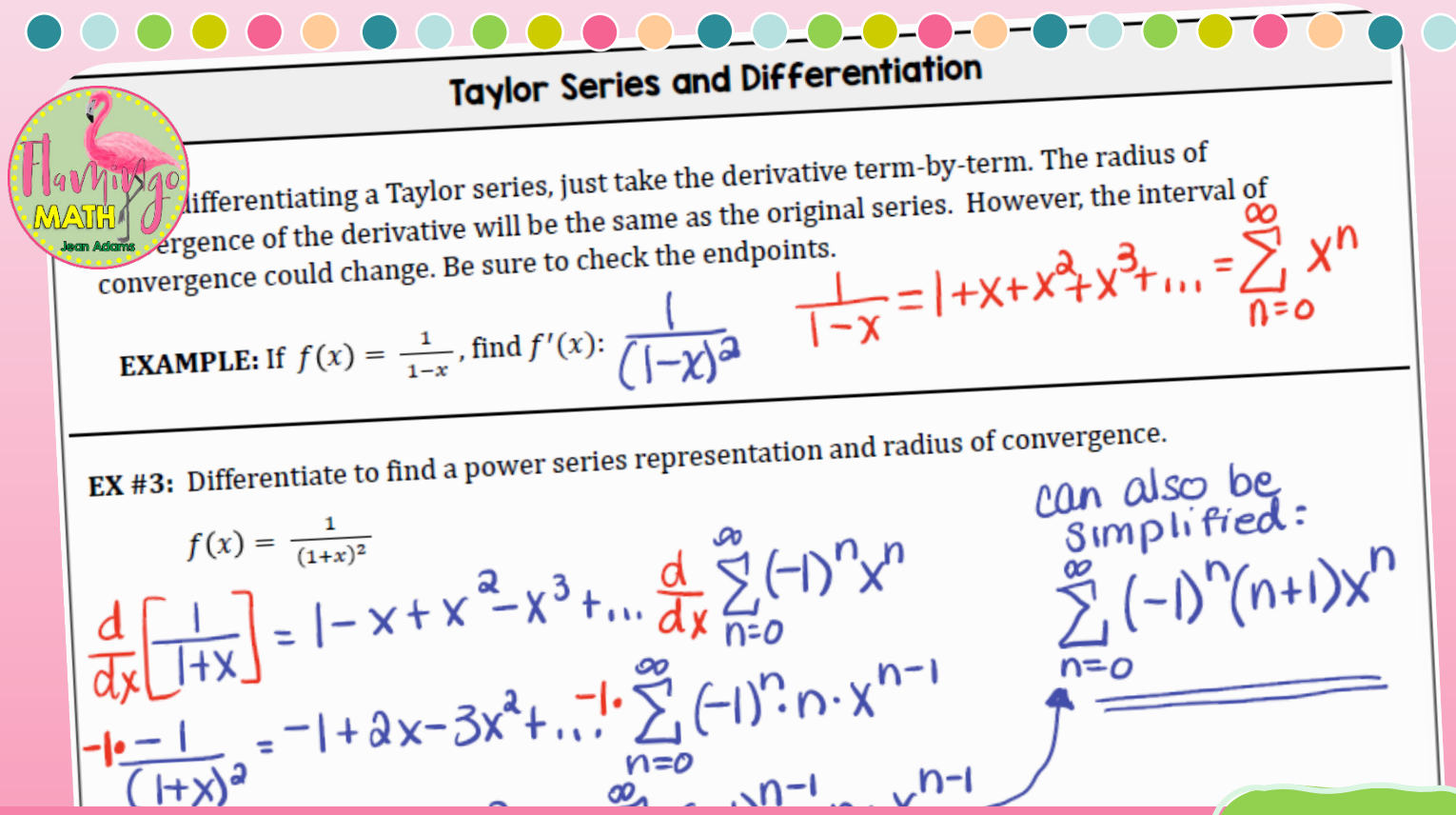
AP® Calculus BC

Taylor and Maclaurin Series



Taylor Series and Differentiation

ifferentiating a Taylor series, just take the derivative term-by-term. The radius of
ergence of the derivative will be the same as the original series. However, the interval of convergence could change. Be sure to check the endpoints.

EXAMPLE: If $f(x) = \frac{1}{1-x}$, find $f'(x)$: $\frac{1}{(1-x)^2}$

$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots = \sum_{n=0}^{\infty} x^n$

EX #3: Differentiate to find a power series representation and radius of convergence.

$f(x) = \frac{1}{(1+x)^2}$

$\frac{d}{dx}\left[\frac{1}{1+x}\right] = 1-x+x^2-x^3+\ldots \quad \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n$

$-1\cdot\frac{-1}{(1+x)^2} = -1+2x-3x^2+\ldots \quad -1\cdot\sum_{n=0}^{\infty}(-1)^n \cdot n \cdot x^{n-1}$

can also be simplified:

$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

Completely Editable

10.9

Lesson 10.8: Power Series and Interval of Convergence

Topic 10.13: Radius and Interval of Convergence of Power Series

We have studied infinite series and learned about Taylor polynomial approximations. In this lesson and the next, we will explore how many functions, such as e^x, $\cos(x)$, $\sin(x)$ can be rewritten as polynomials with an infinite number of terms. **Taylor and Maclaurin series** are actually specific kinds of power series. A power series has a variable in it.

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is called a **power series.** While an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at c,** where c is a constant.

Taylor Polynomials centered about x = c

$$f(x) \approx P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^n(c)}{n!}(x-c)^n$$

EX #1: Some important details to consider. Some fu
Taylor polynomial at $x = c$ because the functio
Determine which functions present problems a

A. $y = \sqrt{x-1}$ centered at $x = 2$

B

C. $y = \cot x$ centered at $x = 0$

D

E. $y = \sqrt{x-1}$ centered at $x = 0$

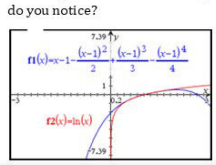
F

There are *two issues* you want to consider. First, we've seen that, except for geometric or telescoping series, we can't sum most series with an infinite number of terms; and also, not all Taylor series will converge for all values of x. But, we will show how **infinite Taylor series (Taylor polynomials with an infinite number of terms) will be exactly equal to their parent functions for x-values that converge.**

EX #2: Given $f(x) = \ln x$

A. Construct a Taylor polynomial of degree 4, centered at $x = 1$.

B. Then use your calculator to compare the original function to your approximation. What do you notice?



d then put it in sigma notation.

ce tests.

39 digits of pi to calculate the circumference of the rogen atom. WOW, now that's a tiny error! So, are this? Well, we're doing this because polynomials he transcendental functions. Often, an accurate e.

☐ **When does a power series converge or diverge?**

Hey, we just spent two weeks on convergence tests in the last unit!

☐ **What test do we use for determining convergence?**

In general, the Ratio Test (sometimes the Root Test) should be used to determine the radius of convergence. Then we can use all of the other tests we learned on the endpoints to determine the interval of convergence.

Radius and Interval of Convergence

A power series in x can be viewed as a function of x

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

where the domain is the set of all x for which the power series converges. Our main concern in this lesson will be to find the domain of a power series. It should be evident that every power series converges at its center, which means c always lies in the domain of f. The domain of a power series will take on three basic forms: a single point, an interval centered at c, or the entire number line.

Convergence of a Power Series

For a power series centered at c, there are **three possibilities:**

1. The series converges only at c.
2. There exists a real number $R > 0$ such that th
   $|x-c| < R$ and diverges for $|x-c| > R$.
3. The series converges absolutely for all x.

- ❑ The number R is the **radius of convergence** c
- ❑ If the series converges only at c, the radius of
- ❑ If the series converges for all x, the radius of c
- ❑ The set of all values of x for which the power s **convergence** of the power series.

In summary, the Ratio Test is used to find the radius says that a series will converge if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, so w determine the value(s) of x for which the limit is less

NOTE: Every time you are asked to find the interv if the endpoints are included in the interva

EX #3: Find the radius of convergence and the inter endpoints.

A. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$

B. $\sum_{n=0}^{\infty} n!\,x^n$

C. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

D. $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

A Brief Summary

- ❑ Many functions can be written as power series $\sum^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{}$

necessary to find the **radius/interval of convergence.**

- ❑ There are some functions that are not equal to the sum of their Taylor series, but we are not responsible for these in our course or on the AP exam.

NOTES

10.9 **Taylor and Maclaurin Series Homework**

Name _____

Date _____ Period _____

Problems 1 – 8, Write the basic Maclaurin series representation, in general form, for each of the following:

1. $T(x)$ centered at $x = c$	**2.** $f(x) = \sin x$
3. $f(x) = \cos x$	**4.** $f(x) = e^x$
5. $f(x) = \dfrac{1}{1-x}$	**6.** $f(x) = \dfrac{1}{1+x}$
7. $f(x) = \tan^{-1} x$	**8.** $f(x) = \ln(1+x)$

9. Find the Maclaurin series for $f(x) = 3e^{-2x}$. Write the first three terms and the general term.

10. Find a Taylor polynomial for $f(x) = \sqrt{1+x}$ of degree 3, centered at $x = 0$.

11. Find a Taylor polynomial of degree $n = 4$ for $f(x) = \ln x$ centered at $c = 1$. Then, use $P_4(x)$ to approximate $\ln(1.1)$.

12. Find the 15th derivative for $f(x) = e^{2x^3}$ at $x = 0$

13. Use a known Maclaurin series to evaluate $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

14. Use a known Maclaurin series to evaluate $\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

15. Express $\int \sin x^2 \, dx$ as a power series.

16. Express $\int \frac{\cos x^2 - 1}{x^2} \, dx$ as a power series.

17. Given $f(x) = x \cos x^3$

A. Find a Maclaurin series, write out the first 3 nonzero terms, and the general term.

B. Find the derivative of $f(x)$, write out the first 3 nonzero terms, and the general term.

18. Let $T_4(x) = 8 - 3(x-2) + 7(x-2)^2 - 4(x-2)^3 + 5(x-2)^4$

A. Find $f(2)$ and $f'''(2)$

B. Write a second-degree Taylor polynomial for $f'(x)$ about $x = 2$ and use it to approximate $f'(2.2)$.

C. Write the fourth-degree Taylor polynomial for $g(x) = \int_2^x f(t)dt$ about $x = 2$.

D. Is it possible to find $f(1)$ using the given information? Explain why or why not.

19. Let $g(x)$ be a function with continuous derivatives and that $g(6) = 2, g'(6) = -3, g''(6) = 1$ and $g'''(6) = -2$.

A. Find a second-degree Taylor polynomial for g about $x = 6$.

Practice Sets

10.9 Taylor & Maclaurin Series **Form A**

Name _____

Date _____Period _____

Given: $f(x) = x^2e^{-2x}$

1. Find a Maclaurin series for f(x). Write the first three nonzero terms and the general term.

2. Find the derivative of $f(x)$ write out the first three nonzero terms and the general term.

10.9 Taylor & Maclaurin Series **Form B**

Name _____

Date _____Period _____

Given: $f(x) = x^3 \cos x$

1. Find a Maclaurin series for f(x). Write the first three nonzero terms and the general term.

2. Find the derivative of $f(x)$ write out the first three nonzero terms and the general term.

Daily Lesson Quiz