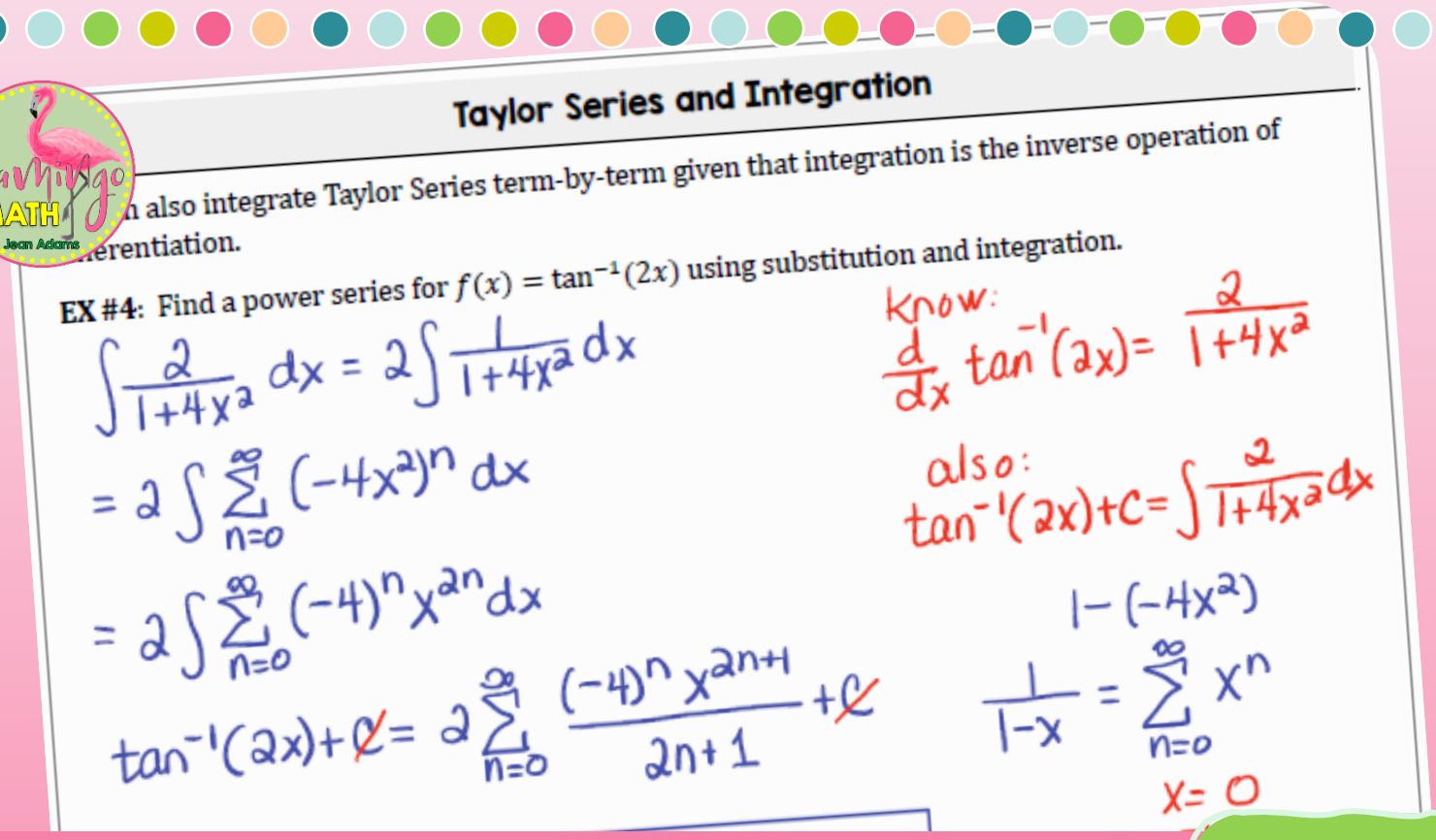
AP® Calculus BC

Representing Functions as Power Series



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10.10

Lesson 10.9: Taylor and Maclaurin Series

Topic 10.14: Finding Taylor or Maclaurin Series for a Function

In the next two lessons, we will investigate operations on Taylor and Maclaurin series and study some techniques and shortcuts for finding a power series that represents a given function.
Let's develop a procedure in general for writing the power series for a given function that will have derivatives of all orders. We will find that *every* ***convergent power series*** *must take on the form:*

Form of a Convergent Power Series

If f is represented by a power series $f(x) = \sum a_n(x-c)^n$ for all x in an open interval containing c, then $a_n = \frac{f^{(n)}(c)}{n!}$ and $f(x) = f(c) + f'(c)(x-c) + \frac{f'(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$

NOTE: Be sure you understand that this theorem says ***if a power series converges to f(x), the series must be a Taylor series.*** It does not say that every series formed with the Taylor coefficients $a_n = f^n(c)/n!$ will converge to $f(x)$. The formal definition of Taylor series is stated below:

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all order at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Is called the **Taylor series for $f(x)$ at c.**

Also, if $c = 0$, then the series is the **Maclaurin series for f.**

Taylor Series Shortcuts

EX #1: Given $f(x) = e^{-2x}$

A. Find a Maclaurin series for $f(x)$. Give the first four nonzero terms and the general term.

B. Use your answer to find a Maclaurin series for $f'(x)$.

EX #2: Evaluate the indefinite integral as a power series and find the radius of convergence.

$$\int \frac{t}{1-t^3}\,dt$$

Taylor Series and Differentiation

When differentiating a Taylor series, just take the derivative term-by-term. The radius of convergence of the derivative will be the same as the original series. However, the interval of convergence could change. Be sure to check the endpoints.

EXAMPLE: If $f(x) = \frac{1}{1-x}$, find $f'(x)$:

EX #3: Differentiate to find a power series representation and radius of convergence.

$$f(x) = \frac{1}{(1+x)^2}$$

Taylor Series and Integration

We can also integrate Taylor Series term-by-term given that integration is the inverse operation of differentiation.

EX #4: Find a power series for $f(x) = \tan^{-1}(2x)$ using substitution and integration.

Binomial Series

The last series we want to develop is for a function of the form $f(x) = (1+x)^k$ which is called the **binomial series.** If k is any number and $|x| < 1$, then

EX #6: Approximate $\int_0^1 x\cos x^3\,dx$ to within three decimal places.

NOTES

10.9 **Taylor and Maclaurin Series Homework**

Name _____

Date _____ Period _____

Problems 1 – 8, Write the basic Maclaurin series representation, in general form, for each of the following:

1. $T(x)$ centered at $x = c$	**2.** $f(x) = \sin x$
3. $f(x) = \cos x$	**4.** $f(x) = e^x$
5. $f(x) = \frac{1}{1-x}$	**6.** $f(x) = \frac{1}{1+x}$
7. $f(x) = \tan^{-1} x$	**8.** $f(x) = \ln(1+x)$

9. Find the Maclaurin series for $f(x) = 3e^{-2x}$. Write the first three terms and the general term.

10. Find a Taylor polynomial for $f(x) = \sqrt{1+x}$ of degree 3, centered at $x = 0$.

11. Find a Taylor polynomial of degree $n = 4$ for $f(x) = \ln x$ centered at $c = 1$. Then, use $P_4(x)$ to approximate $\ln(1.1)$.

12. Find the 15th derivative for $f(x) = e^{2x^3}$ at $x = 0$

13. Use a known Maclaurin series to evaluate $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

14. Use a known Maclaurin series to evaluate $\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

15. Express $\int \sin x^2 \, dx$ as a power series.

16. Express $\int \frac{\cos x^2 - 1}{x^2} \, dx$ as a power series.

17. Given $f(x) = x \cos x^3$

A. Find a Maclaurin series, write out the first 3 nonzero terms, and the general term.

B. Find the derivative of $f(x)$, write out the first 3 nonzero terms, and the general term.

18. Let $T_4(x) = 8 - 3(x-2) + 7(x-2)^2 - 4(x-2)^3 + 5(x-2)^4$

A. Find $f(2)$ and $f'''(2)$

B. Write a second-degree Taylor polynomial for $f'(x)$ about $x = 2$ and use it to approximate $f'(2.2)$.

C. Write the fourth-degree Taylor polynomial for $g(x) = \int_2^x f(t)dt$ about $x = 2$.

D. Is it possible to find $f(1)$ using the given information? Explain why or why not.

19. Let $g(x)$ be a function with continuous derivatives and that $g(6) = 2, g'(6) = -3, g''(6) = 1$ and $g'''(6) = -2$.

A. Find a second-degree Taylor polynomial for g about $x = 6$.

Practice Sets

10.9 Taylor & Maclaurin Series **Form A**

Name _____

Date _____Period _____

Given: $f(x) = x^2e^{-2x}$

1. Find a Maclaurin series for f(x). Write the first three nonzero terms and the general term.

2. Find the derivative of $f(x)$ write out the first three nonzero terms and the general term.

10.9 Taylor & Maclaurin Series **Form B**

Name _____

Date _____Period _____

Given: $f(x) = x^3 \cos x$

1. Find a Maclaurin series for f(x). Write the first three nonzero terms and the general term.

2. Find the derivative of $f(x)$ write out the first three nonzero terms and the general term.

Daily Lesson Quiz