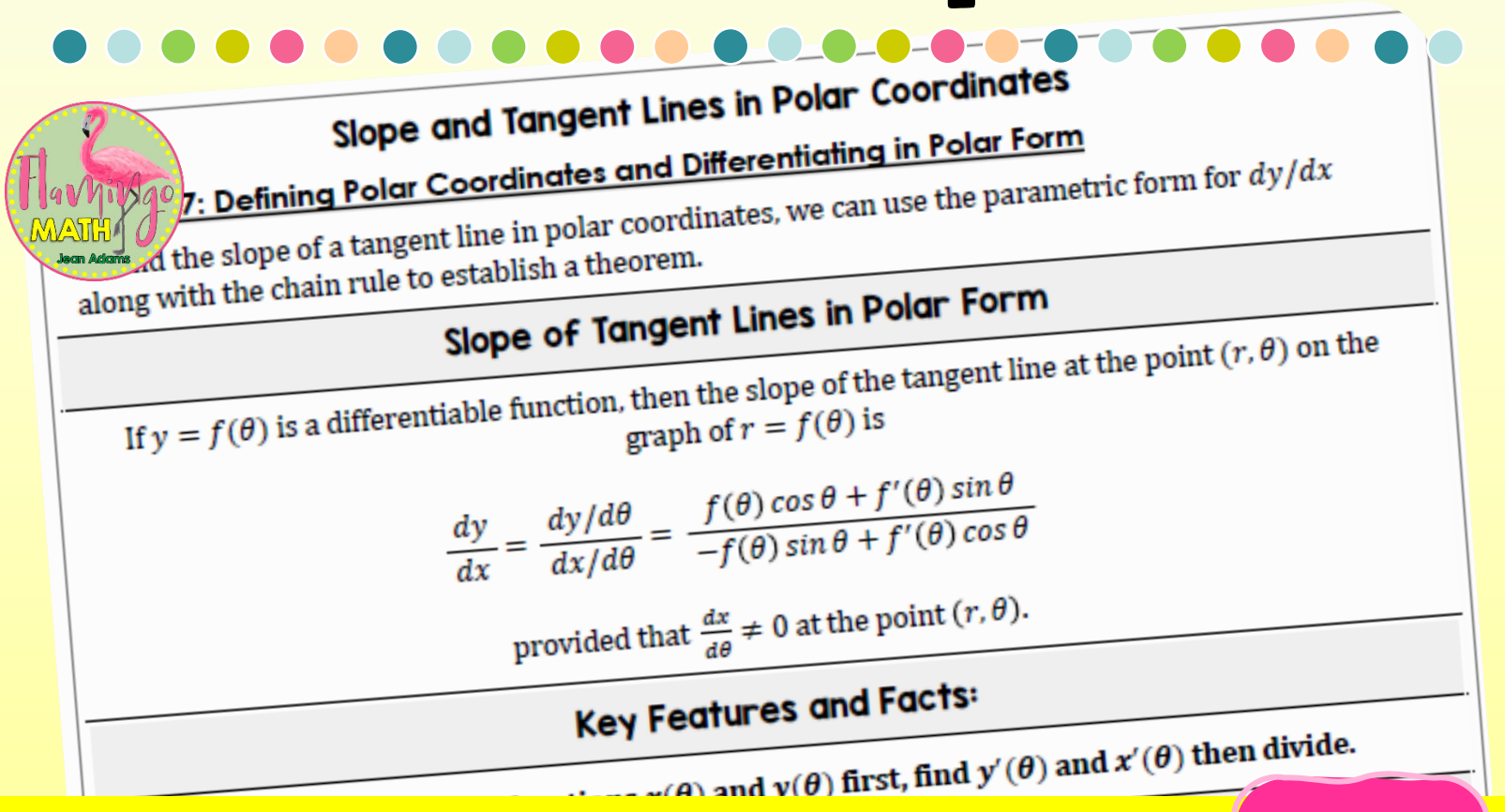
AP® Calculus BC

# Polar Coordinates and Graphs



## Slope and Tangent Lines in Polar Coordinates

### 9.7: Defining Polar Coordinates and Differentiating in Polar Form

…d the slope of a tangent line in polar coordinates, we can use the parametric form for $dy/dx$ along with the chain rule to establish a theorem.

### Slope of Tangent Lines in Polar Form

If $y = f(\theta)$ is a differentiable function, then the slope of the tangent line at the point $(r, \theta)$ on the graph of $r = f(\theta)$ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $\frac{dx}{d\theta} \neq 0$ at the point $(r, \theta)$.

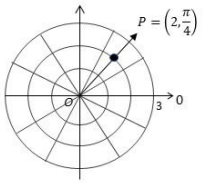
### Key Features and Facts:

…$x(\theta)$ and $y(\theta)$ first, find $y'(\theta)$ and $x'(\theta)$ then divide.

**Completely Editable** 9.5

# Lesson 9.5 Polar Coordinates and Graphs

While we commonly use the **rectangular** (Cartesian Coordinate System) coordinates as our graphing tool of choice, we can show that some problems in mathematics are easier to analyze in a polar coordinate system. The Cartesian plane allows us to identify any point in the plane by a set of numbers, such as $(x, y)$ or $(x, y, z)$. The equations corresponding to these graphs have been in either rectangular form or parametric form.




In **polar coordinates** a point in the plane is defined by a pair of numbers $(r, \theta)$. The number $\theta$ measures the angle between the positive $x$-axis, called the **polar axis**, and a ray that passes through the point. The number $r$ measures the directed distance from the **pole** (or **origin**) $O$ to the point $P$. Now you can think of the graph of the polar equation $r = f(\theta)$ as the graph of the parametric equations $x = f(t)\cos t$, $y(t) = f(t)\sin t$ where $t = \theta$ and:

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

In this lesson and the next, we will explore the polar coordinate system, conversion of equations, graphs and focus on tangent lines, arc length, and area.

## A Quick Crash Course for Polar Coordinates

The polar coordinates $(r, \theta)$ are related to the rectangular coordinates $(x, y)$ as follows:

Just like converting points from one system to another, we can also convert equations.

## Polar Equations

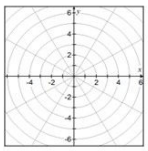
**EX #3:** Convert the rectangular equations to a polar equation. Then, verify with your calculator.

| | | |
|---|---|---|
| **A.** $y = 4$ | **B.** $y = 3x + 2$ | **C.** $y = 2x^2$ |

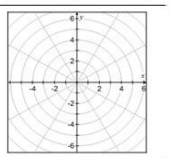
**EX #4:** Convert the polar equations to a rectangular equation. Then, verify with your calculator.

| | | |
|---|---|---|
| **A.** $r = -3\sec\theta$ | **B.** $r = \frac{4}{1 + 2\cos\theta}$ | **C.** $r = \frac{3}{\cos\theta + 2\sin\theta}$ |

**B.** $r = -3\sin\theta$



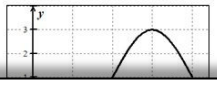
**C.** $r = \frac{5}{\cos\theta}$



Sketching polar graphs by hand can be quite tedious. You can use an *"auxiliary graph"* to help guide you.

## Using Auxiliary Graphs to Sketch Polar Curves

**EX #6:** Consider the function $y = 1 - 2\sin x$ for $0 \le x \le 2\pi$, shown at right; and, its relationship to the polar graph $r = 1 - 2\sin\theta$. Since there will be unique $y$-values on this domain, we can use the table below to assist in graphing the polar graph. Consider the values for $(x, y)$




## Slope and Tangent Lines in Polar Coordinates

Topic 9.7: Defining Polar Coordinates and Differentiating in Polar Form

To find the slope of a tangent line in polar coordinates, we can use the parametric form for $dy/dx$ along with the chain rule to establish a theorem.

## Slope of Tangent Lines in Polar Form

If $y = f(\theta)$ is a differentiable function, then the slope of the tangent line at the point $(r, \theta)$ on the graph of $r = f(\theta)$ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $\frac{dx}{d\theta} \neq 0$ at the point $(r, \theta)$.

## Key Features and Facts:

**NOTE: You can create the functions $x(\theta)$ and $y(\theta)$ first, find $y'(\theta)$ and $x'(\theta)$ then divide.**

| Horizontal Tangents | Vertical Tangents | No Conclusion |
|---|---|---|
| Find where $\frac{dy}{d\theta} = 0$ provided $\frac{dx}{d\theta} \neq 0$ | Find where $\frac{dx}{d\theta} = 0$ provided $\frac{dy}{d\theta} \neq 0$ | If both $\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$ |

**EX #7:** Find the points at which the cardioid given by $r = 1 + \cos\theta$ has vertical or horizontal

## Tangent Lines at the Pole

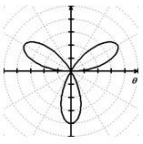
If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

**NOTE:** If you find the zeros of $r = f(\theta)$ they can help you find the tangent lines at the pole. In the next example, the three-leafed rose curve has more than one tangent line at the pole.

**EX #8:** Given the rose curve: $r = 4\sin 3\theta$

**A.** Find $\frac{dy}{dx}$

**B.** Calculate the slopes of the three tangent lines for the curve at the pole. $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$

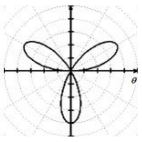


**C.** Use a graphing utility to find the locations of all horizontal tangent lines. Discuss and significance of these points.

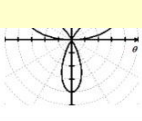


Graph of $y = dy/d\theta$

**D.** Find the three points where |r| is a maximum. Show that the tangent line is perpendicular to the line segment connecting the point to the origin.



**E.** Write the equation for the tangent line that passes through the point $(r, \theta) = \left(4, \frac{\pi}{6}\right)$



NOTES

**9.5** **Polar Coordinates and Graphs Homework**

Name _____

Date _____ Period _____

**Problems 1 – 2,** Find the equivalent rectangular coordinates for each pair of polar coordinates.

**1.** $\left(\sqrt{3}, \frac{\pi}{6}\right)$

**2.** $\left(-5, \frac{5\pi}{3}\right)$

**Problems 3 – 4,** Find an equivalent pair of polar coordinates for each rectangular coordinate pair.

**3.** $(4, -4)$

**4.** $\left(-3, -\sqrt{3}\right)$

**Problems 5 - 7 ,** Convert the following equations to polar form.

**5.** $x = 2$

**6.** $x^2 + y^2 = 7$

**7.** $3x - 2y = 5$

**Problems 8 – 10,** Convert the following equations to rectangular form.

**8.** $r = 3\csc\theta$

**9.** $r = 3\cos\theta$

**10.** $r = 4\tan\theta\sec\theta$

**Problems 11 – 12,** For each of the following, find $dy/dx$ for the given value of $\theta$.

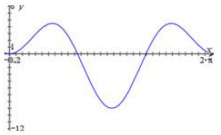
**11.** $r = 3\sin\theta, \theta = \frac{\pi}{3}$

**12.** $r = 2 + 3\sin\theta, \theta = \frac{\pi}{2}$

**Problems 13 – 14,** For each of the following, find the polar points of horizontal and vertical tangency, if any.

**13.** $r = 2\sin(\theta)$ **on** $[0, \pi]$

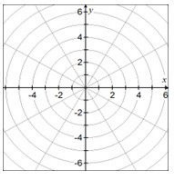
**14.** $r = 4(1 - \cos\theta)$ **on** $[0, 2\pi]$



Graph of $y = dy/d\theta$

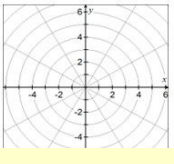
**Problems 15 – 16,** Sketch a graph of the polar equation and find the tangents at the pole.

**15.** $r = 4\cos(3\theta), 0 \le \theta \le \pi$



**16.** Given: $r = 2 - 4\sin\theta, \ 0 \le \theta \le 2\pi$

**A.** Make a table and sketch the graph



# Practice Sets

9.5 Polar Coordinates and Graphs **Form A**

Name _____

Date _____Period _____

Given the graph of the polar curve:

$r = 3 - 3\sin\theta$

1. Find $\frac{dy}{dx}$

2. Find the slope of the tangent line at the point $(3,\ 2\pi)$

3. Find the polar points of horizontal tangent lines.



9.5 Polar Coordinates and Graphs **Form B**

Name _____

Date _____Period _____

Given the graph of the polar curve:

$r = 2 + 2\cos\theta$

1. Find $\frac{dy}{dx}$

2. Find the slope of the tangent line at the point $(4,\ 2\pi)$

3. Find the polar points of horizontal tangent lines.



# Daily Lesson Quiz