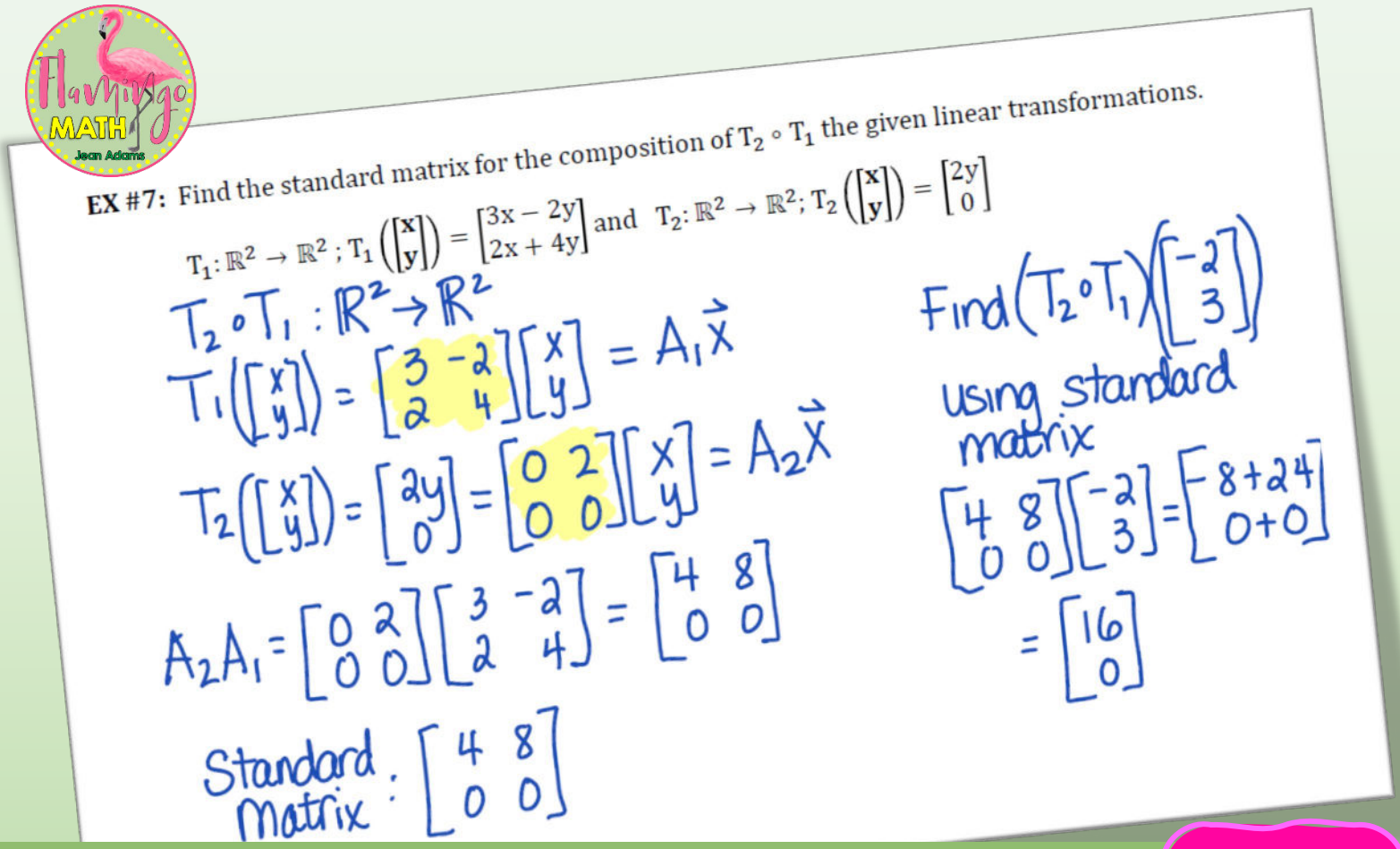
AP® Precalculus

Matrices as Functions



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4.13

Lesson M-A Matrices as Functions

In this lesson we want to determine the association between a linear transformation and a matrix.. Recall that we can multiply an $m \times n$ matrix by an $n \times 1$ column vector which will result in an $m \times 1$ column vector. The $n \times 1$ vector belongs to $\mathbb{R}^n$, which is the set of all $n \times 1$ vectors.

The Linear Transformation Mapping

The linear transformation mapping $\langle x, y\rangle$ to $\langle a_{11}x + a_{12}y + a_{21}x + a_{22}y\rangle$ is

associated with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

EX #1: Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}$. Show that by matrix multiplication A transforms vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$. We need a 3×1 matrix to produce the 2×1 result.

A. Use the column matrix $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ for a numeric example.

B. Use the column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for a general example.

The example above demonstrates a transformation achieved by matrix multiplication. The notation $T(\vec{x})$ means the transformation T applied to the vector $\vec{x}$. We can write $T_A(\vec{x}) = A\vec{x}$.
That is, T_A is the transformation determined by the matrix A. This is what we mean when we say that *A transforms vectors*. In this case we say that *T is a matrix transformation.*

Remember the property of matrix multiplication given scalars c and d:
$A(c\vec{\mathbf{u}} + d\vec{\mathbf{v}}) = cA\vec{\mathbf{u}} + dA\vec{\mathbf{v}}$

For matrix A an $m \times n$ matrix and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, $n \times 1$ vectors in $\mathbb{R}^n$, this property holds.
That is, matrix multiplication gives an example of a linear transformation.

Linear Transformation Definition

Let $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function, where each $\vec{\mathbf{x}} \in \mathbb{R}^n$, $T(\vec{\mathbf{x}}) \in \mathbb{R}^m$. Then T is a **linear transformation** if whenever scalars c and d and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are $n \times 1$ vectors in $\mathbb{R}^n$,

$$T(c\vec{\mathbf{u}} + d\vec{\mathbf{v}}) = cT(\vec{\mathbf{u}}) + dT(\vec{\mathbf{v}})$$

Elementary Vectors

The vectors along the standard rectangular coordinate axes of $\mathbb{R}^2$ are denoted

$\mathbf{e_1} \coloneqq \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 \coloneqq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The are called **elementary vectors** (hence notation $\mathbf{e}_i$, I =1,2), and the ordered list $(\mathbf{e}_1, \mathbf{e}_2)$ is called the **standard basis** of $\mathbb{R}^2$.

Matrix Transformations are Linear Transformations

Given a linear map $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ we will say that an $m \times n$ matrix A is a *matrix representing the linear transformation* T if the image of a vector $\mathbf{x}$ in $\mathbb{R}^n$ is given by the matrix vector product $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$, then T is a linear transformation.

EX #2: Find the standard matrix A for the dilation transformation $T(\mathbf{x}) = 4\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^2$.

Then , find the $|\det A|$. What does this value tell you about the dilation in $\mathbb{R}^2$ under T?

EX #3: The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -4 \\ 6 \\ -1 \end{bmatrix}$.
Write a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

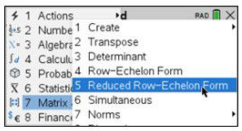
Finding a Vector Using Technology

EX #4: Let $T(\mathbf{x}) = A\mathbf{x}$ find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$

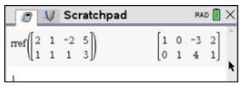
A. Given : $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

B. Determine whether $\mathbf{x}$ is unique.

With the TI -Nspire CX CAS click
1. MENU>7:>5> MENU>7>1>1
2. Set the matrix as 2 Rows and 4 Columns



3. Enter the elements from A and b in Row 1 and Row 2



4. Solve the Reduced Row Echelon Form for x_1 and x_2

Counterclockwise Rotation Matrix

The matrix associated with a linear transformation of vectors that maps every vector to the vector that is an angle θ counterclockwise rotation about the origin from the original vector is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

EX #5: Given the vector $\langle 3,4\rangle$ rotate counter-clockwise $\frac{\pi}{2}$

Lesson M-B: Matrices as Functions

Recall that we can take the composition of function f and g:
$(f \circ g)x) = f(g(x))$ and $(g \circ f)x) = g(f(x))$
This same operation holds true for linear transformations of matrices if the domains and codomains match up.

Composition of Linear Transformations

Let $T_1: \mathbf{U} \to \mathbf{V}$ and $T_2: \mathbf{V} \to \mathbf{W}$ be linear transformations.

Then the **composition of T_2 with T_1** is the linear transformation

$(T_2 \circ T_1): \mathbf{U} \to \mathbf{W}$

defined by $(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$

for all $\vec{\mathbf{u}} \in \mathbf{U}$.

NOTE: If the range of T_1 is **not** contained in the domain T_2, then $(T_2 \circ T_1)$ is **not well-defined**.

The order of compositions matter.

EX #6: Given linear transformations $T_1: \mathbb{R}^2 \to \mathbb{R}^3$ and $T_2: \mathbb{R}^3 \to \mathbb{R}^2$, find $(T_2 \circ T_1)\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$

where $T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ 2x_1 \end{bmatrix}$ and $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_3 \\ 2x_2 - 4x_3 \end{bmatrix}$

Then the product of the standard matrices A_2A_1 is the standard matrix for linear transformation

Then the **composition of T_2 with T_1** is the linear transformation

$(T_2 \circ T_1): \mathbf{U} \to \mathbf{W}$

Guided Notes

EX #7: Find the standard matrix for the composition of $T_2 \circ T_1$ the given linear transformations.

$T_1: \mathbb{R}^2 \to \mathbb{R}^2$; $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x - 2y \\ 2x + 4y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$; $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2y \\ 0 \end{bmatrix}$

EX #8: Determine the transformation matrix that describes a reflection in the y-axis followed by a rotation of $\frac{\pi}{2}$ about the origin and a dilation by a factor of 3.

Lesson 14-C: Matrices as Functions

Finally, let's look at the possibilities of two linear transformations having an inverse. Remember that a function f has an inverse, we call the inverse f^{-1}, and these two functions "undo each other. Formally, we say $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$ for all values o x in the domain of f(this is also the range of f^{-1}). We can do the same thing for linear transformations of matrices.

Inverse of a Linear Transformation

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$.

- ❑ T is invertible if and only if A is invertible.
- ❑ If T is invertible, its standard matrix is A^{-1}.

EX #9: Describe the inverse of the following transformations in the plane:

A. A rotation of $\frac{\pi}{6}$ counterclockwise

B. A horizontal contraction of $k = 0.25$

C. A reflection across the line $y = 2x$

EX #10: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be dilation by a factor of 3, that is $T(\mathbf{x}) = 3\mathbf{x}$. If T is invertible, find T^{-1}.

EX #11: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection over the y-axis. Is $T(\mathbf{x})$ invertible? If so, find T^{-1}.

EX #12: Find the inverse of this linear transformation, if it exists, and write in the same form.

$T(x, y, z) = (y + 2z, x + y + z, 3z - x)$

EX #13: Find T^{-1} for the linear transformation. Where $T: \mathbb{P}_2 \to \mathbb{R}^3$.

$T(ax^2 + bx + c) = (a + c, 2b, a - c)$

Guided Notes

4.13 **Matrices as Functions Practice Set 1**

Name _____

Date _____ Period _____

Problems 1-2, T is defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$.

1. $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

Problems 3-4, Find a formula for $T(\vec{\mathbf{x}})$ for any $\vec{\mathbf{x}}$ in $\mathbb{R}^2$.

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \\ -7 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$

4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$

6. Given: $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 4 \\ 3 & -2 & -6 \end{bmatrix}$, $b = \begin{bmatrix} -4 \\ 3 \\ -6 \end{bmatrix}$ Determine whether $\mathbf{x}$ is unique.

Problems 7-8, Complete the missing entries of the matrix. Assume that the equation holds for all values of the variables.

7. $\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 4y \\ -5x + y \\ y \end{bmatrix}$

8. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + 2x_3 \\ 4x_2 - 5x_3 \\ x_1 - x_2 + x_3 \end{bmatrix}$

Problems 9-10, Use rotation matrices to find the following standard matrix for T.

4.13 **Matrices as Functions Practice Set 2**

Name _____

Date _____ Period _____

Problems 1-4, find the standard matrix for the composition of $T_2 \circ T_1$ for the given linear transformations.

1. $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$; $T_1\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x + 3y \end{bmatrix}$ and $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$; $T_2\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \end{bmatrix}$

2. $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$; $T_1\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2x + y \end{bmatrix}$ and $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$; $T_2\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ x \end{bmatrix}$

Problems 5-6, find the standard matrix for the composition of $T_1 \circ T_2$ for the given linear transformations.

5. $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$; $T_1\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2x + y \end{bmatrix}$ and $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$; $T_2\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ x \end{bmatrix}$

6. $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$; $T_1\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) = \begin{bmatrix} 2x \\ -3y \end{bmatrix}$ and $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$; $T_2\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ 2x \end{bmatrix}$

Problems 7- 8, find the transformation matrix for the following.

4.13 **Matrices as Functions Practice Set 3**

Name _____

Date _____ Period _____

Problems 1-4, Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation described. If $T(\mathbf{x})$ is invertible, find T^{-1}.

1. A contraction by a factor of 0.75 $T(\mathbf{x}) = 0.75\mathbf{x}$

2. A reflection through the line $y = x$ where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. A reflection through the origin, where $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

4. A rotation of $\frac{\pi}{3}$ clockwise.

Problems 5-8, find the inverse T^{-1} for each linear transformation $T(\mathbf{x})$

5. $T(\mathbf{x}, \mathbf{y}) = (3x - 5y, -4x + 8y)$

6. $T(\mathbf{x}, \mathbf{y}) = (2x + 7y, 3x - 4y)$

Problems 9-10, Find the inverse of each linear transformation, if it exists, and write in the same form.

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation described by $T(\mathbf{x}, \mathbf{y})$. If $T(\mathbf{x})$ is invertible, find T^{-1}.

A. Find T^{-1}

B. Find A, use that to find A^{-1}, then interpret as a linear transformation.

C. Verify that it is an inverse by finding $T^{-1}(T(\mathbf{x}, \mathbf{y}))$

9. $T(x, y) = (3x - 5y, x - 2y)$

10. $T(x, y) = (4x + 5y, 2x + 3y)$

Practice Sets

4.13 Matrices as Functions Set 1 **Form A**

Name _____

Date _____ Period _____

For problems 1 and 2
T is defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$. Is it unique?

1. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 4 & 1 & -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 9 \\ -3 \\ 3 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

3. Complete the missing entries of the matrix. Assume that the equation holds for all values of the variables.

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x - 2y + z \\ x + 5z \\ 2y - 3z \end{bmatrix}$$

Grade:

4.13 Matrices as Functions Set 1 **Form B**

Name _____

Date _____ Period _____

For problems 1 and 2
T is defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$. Is it unique?

1. $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 2 \\ 3 & -1 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ -14 \\ -6 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

3. Complete the missing entries of the matrix. Assume that the equation holds for all values of the variables.

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x - z \\ x + 2y - 5z \\ 2y + 3z \end{bmatrix}$$

Grade:

4.13 Matrices as Functions Set 2 **Form A**

Name _____

Date _____ Period _____

Find the standard matrix for the composition of $T_2 \circ T_1$ for the given linear transformations.

1. $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x - y \\ x + 2y \end{bmatrix}$ and $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 3y \end{bmatrix}$

2. $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3x - y \end{bmatrix}$ and $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 5y \\ x \end{bmatrix}$

3. $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ 3y - x \end{bmatrix}$ and $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2y \end{bmatrix}$

Grade:

4.13 Matrices as Functions Set 2 **Form B**

Name _____

Date _____ Period _____

Find the standard matrix for the composition of $T_2 \circ T_1$ for the given linear transformations.

1. $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ x + y \end{bmatrix}$ and $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 4y \end{bmatrix}$

2. $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x - 2y \end{bmatrix}$ and $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 3y \\ x \end{bmatrix}$

3. $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ 2y - x \end{bmatrix}$ and $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3y \end{bmatrix}$

Grade:

4.13 Matrices as Functions Set 3 **Form A**

Name _____

Date _____ Period _____

1. If $T(\mathbf{x})$ is invertible, find T^{-1}. A rotation of $\frac{\pi}{6}$ clockwise.

Problems 2 and 3, find the inverse for each linear transformation.

2. $T(x, y) = (2x - 3y, -5x + 2y)$

Grade:

4.13 Matrices as Functions Set 3 **Form B**

Name _____

Date _____ Period _____

1. If $T(\mathbf{x})$ is invertible, find T^{-1}. A rotation of $\frac{\pi}{4}$ clockwise.

Problems 2 and 3, find the inverse for each linear transformation.

2. $T(x, y) = (4x + 3y, 2y)$

Grade:

Daily Lesson Quiz