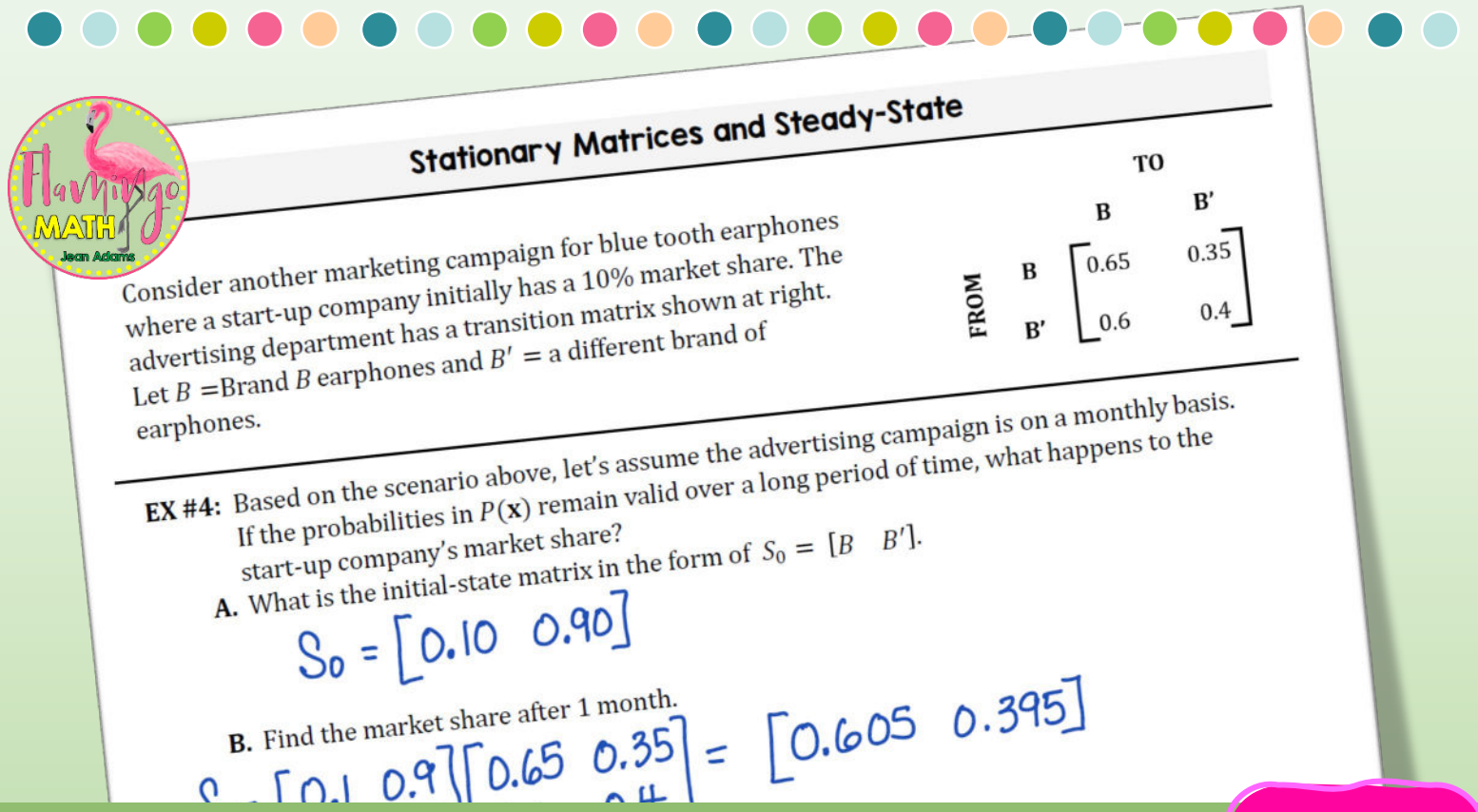
AP® Precalculus

Matrices Modeling Contexts



Stationary Matrices and Steady-State

Consider another marketing campaign for blue tooth earphones where a start-up company initially has a 10% market share. The advertising department has a transition matrix shown at right. Let B =Brand B earphones and B' = a different brand of earphones.

FROM \ TO	B	B′
B	0.65	0.35
B′	0.6	0.4

EX #4: Based on the scenario above, let's assume the advertising campaign is on a monthly basis. If the probabilities in $P(\mathbf{x})$ remain valid over a long period of time, what happens to the start-up company's market share?

A. What is the initial-state matrix in the form of $S_0 = [B \quad B']$.

$S_0 = [0.10 \quad 0.90]$

B. Find the market share after 1 month.

$[0.1 \quad 0.9]\begin{bmatrix} 0.65 & 0.35 \\ 0.6 & 0.4 \end{bmatrix} = [0.605 \quad 0.395]$

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Lesson 14-A: Matrices Modeling Contexts

In this final lesson we will turn our attention to application problems using matrices. You will learn about the **Markov Chain** or **Markov Model**, named after the mathematician Andrey Markov in the early 1900's. The Markov decision process provides a mathematical framework for modeling decision making in situations where outcomes are partly random and partly under control of the decision maker. At each time step, the process is in some state S. The decision maker may choose an action A that is available. The process randomly moves into a new state S' which gives the decision maker a reward for their choice.
The probability that the process moves to the new state S' is influenced by the chosen action. $P_A(S, S')$. This process is useful in optimization problems and often appear in biology, ecology, manufacturing, business, and finance.

Transition Matrices

A transition matrix is used to organize information from an arrow diagram. Each row in the matrix represents an initial state. Each column represents a terminal state. For the example at right, Row 1 Column 2 shows the probability that an "action" from process A moving to process B at the next "time step" is 20%. Notice that each row adds to 1.

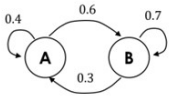
FROM \ TO	A	B
A	0.8	0.2
B	0.3	0.7

$\mathbf{T}_{ij} = P(\text{next state is the state in column } j \mid \text{current state is the state in row } i$

EX #1: We can draw arrow diagrams to represent some process.

Suppose two companies are in competition against each other. The arrow diagram at right explains the percentage for customers switching from Company A to Company B.

Draw a **transition matrix** to represent the data.



EX #2: The transition matrix for a tennis player is shown at right. Let **C** represent the cross-court shots and **D** represents shots down-the-line.

A. If the player hits the first shot down-the-line, what is the probability he will hit a cross-court shot?

B. Explain the meaning of row 2, column 2 in context.

Current Shot \ Next Shot	C	D
C	0.88	0.12
D	0.90	0.10

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The Power of Advertising

Let's consider a marketing campaign for your new protein drink called **Pro Plus.** Your company currently controls 15% of the market share for these protein drinks. Your marketing research predicts that an aggressive advertising campaign will have customer's switching to your Pro Plus drink from a competitor's drink with 60% probability. They also predict that current customers will stay with **Pro Plus** with 90% probability. This research is based on a customer's weekly purchase of protein drinks in a localized area.

EX #3: Based on the scenario above, let A = Pro Plus Drink and let A' = Any other protein drink.

A. Draw an arrow diagram to represent the current state.

B. Create a transition probability matrix to represent the arrow diagram.

C. The market share for your drink is 15%. We want to write an initial state distribution matrix in the form of $S_0 = [A \quad A']$.

D. You want to calculate the probability for one week after the ad campaign runs. Use matrix multiplication to find $S_1 = S_0 \cdot P(\mathbf{x})$, assuming the probability matrix remains constant. Explain your increase in market share.

E. Is there improvement after week 2? Show this calculation. What is your market share for **Pro Plus** after the second week?

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Stationary Matrices and Steady-State

Consider another marketing campaign for blue tooth earphones where a start-up company initially has a 10% market share. The advertising department has a transition matrix shown at right. Let B =Brand B earphones and B' = a different brand of earphones.

FROM \ TO	B	B'
B	0.65	0.35
B'	0.6	0.4

EX #4: Based on the scenario above, let's assume the advertising campaign is on a monthly basis. If the probabilities in $P(\mathbf{x})$ remain valid over a long period of time, what happens to the start-up company's market share?

A. What is the initial-state matrix in the form of $S_0 = [B \quad BB']$.

B. Find the market share after 1 month.

C. What does this tell you about the market share after 1 month?

D. Calculate the findings for S_2, S_3, S_4, assuming the probability matrix remains constant. Use the capabilities of a graphing utility to calculate your results.

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Lesson 14-B: Matrices Modeling Contexts

In the previous example we saw the market share for Bluetooth earphones leveled off after a period of time using the recommended advertising campaign. This matrix is called a **stationary matrix,** and the system is said to be at **steady-state** or **state of equilibrium.** Does every Markov chain have a unique stationary matrix? Will each successive state matrix always approach this stationary matrix? Well, the answer is ***yes*, for a regular Markov chain.**

Regular Markov Chains

A transition matrix P is regular if some power of P has only positive entries. A Markov chain is a regular Markov chain if its transition matrix is regular.

Properties of Regular Markov Chains

Let P be the transition matrix for a regular Markov chain,

- ❑ There is a unique stationary matrix S that can be found by solving the equation $S \cdot P(\mathbf{x}) = S$
- ❑ Given any initial-state matrix S_0, the state matrices S_k approach the stationary matrix S.
- ❑ The matrices P^k approach a limiting matrix $\overline{P}$, where each row of $\overline{P}$ is equal to the stationary matrix S.

EX #5: Which of the following matrices are regular?

A. $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$ **B.** $P = \begin{bmatrix} 0.3 & 0.7 \\ 1 & 0 \end{bmatrix}$ **C.** $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

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Guided Notes

EX #6 [continued]: Suppose the transition matrix for a Markov Chain is $P = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$.

B. Find the stationary matrix S by solving $S \cdot P(\mathrm{x}) = S$

C. Does your result from part [B] match the result from [A]?

Application of Regular Markov Chains

EX #7: **Brightline**© is the new Florida high-speed train service from Miami to Orlando. In the first month of operation, it is expected that 15% of commuters are using the system in the Miami vicinity, while 85% still travel by automobile. The transition matrix below was determined from records of other large rail systems.

A. What is the initial-state matrix?

Current month:		Next month: Train	Next month: Auto
	Train	0.6	0.4
	Auto	0.3	0.7

B. What percentage of commuters will be using the new high-speed train after 1 month? 2 months?

C. What percentage of commuters are using each type of transportation after the high-speed train has been in service for an extended period of time?

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Regular Markov Chains

A Markov chain is said to be a Regular Markov chain if some power of it has only positive entries. Let **T** be a transition matrix for a regular Markov chain.

1. As we take higher powers of **T**, $\mathbf{T}^n$, as n becomes large, approaches a state of equilibrium.
2. If S_0 is any distribution vector, and E an equilibrium vector, then $S_0 \cdot T(\mathbf{x}) = E$
3. Each row of the equilibrium matrix $\mathbf{T}^n$ is a unique equilibrium vector E such that $ET = E$.
4. The equilibrium distribution vector E can be found by letting $ET = E$.

Predicting Past States

The product of the inverse of a matrix that models transitions between states and a corresponding state vector can predict past states. Recall from previous lessons when solving matrix equations in the form $A\mathbf{X} = b$, we can find a solution for A by using the inverse matrix where $A = b \cdot \mathbf{X}^{-1}$

EX #8: Consider that you know the transition matrix for a widget that your company produces. You also know the state vector for one month of an advertising campaign.
$S_1 = [0.3 \quad 0.7]$.

Can you find the initial market share for your company using inverse matrix operations?

Current month:		Next month: A	Next month: A'
	A	0.6	0.4
	A'	0.2	0.8

A. Find $T^{-1}(\mathbf{x})$.

B. Solve the matrix equation for the distribution vector, $S_0 = b \cdot T^{-1}(\mathbf{x})$

C. Explain the initial market share of your product based on your calculations.

D. What is the steady-state or long-term market share for the company? Explain.

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Lesson 14-C: Matrices Modeling Contexts

In this final section, let's look at some application problems where the transition matrix is a 3 × 3 regular Markov chain.

Long-Term Predictions

EX #9: Let's say three companies, A, B, and C, are in competition with each other. We know the transition matrix **T** for customers switching each month among them is given by the matrix below.
If the initial market share for companies A, B, and C is [0.25 0.30 0.45]

A. What is the long-term distribution of market share for each company?

Current month:		Next month: A	Next month: B	Next month: C
	A	0.2	0.3	0.5
	B	0.6	0.2	0.2
	C	0.1	0.3	0.6

EX #10: In a bikeshare program of a large city, there are 3 stations where people can borrow a bike or scooter and return it to any of the 3 stations.
The transition matrix for this Markov process is shown at right.

From Station:		To Station: A	To Station: B	To Station: C
	A	0.3	0.2	0.5
	B	0.1	0.6	0.3
	C	0.2	0.1	0.7

A. If the initial distribution of bikes and scooters is 40% at station A, 25% at station B, and 35% at station C, what will be the distribution after 2 days?

the eventual long-term distribution be different than the initial distribution given in part [A]?

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One Last Application

EX #11: Let's say three department stores, A, B, and C are in competition with each other. We know the following about shopping and market distributions:

- 10% of shoppers at store A will switch to store B every month and 10% to store C.
- 10% of store B shoppers switch to store A and 20% to store C.
- 20% of store C shoppers switch to store A and 20% to store C.

A. What is the transition matrix?

B. What is the stable distribution matrix?

C. What will be the eventual long-term distribution percentage for shoppers in each department store?

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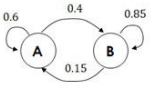
4.14 **Matrices Modeling Contexts Practice Set 1**

Name _____

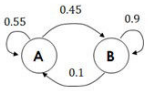
Date _____ Period _____

Problems 1-2, The arrow diagrams shown below explain the percentage for customers switching from Company A to Company B. Draw a **transition matrix** to represent the data.

1.



2.



Problems 3-4, Use the transition matrix $T(\vec{x})$ for each scenario below.

3. Your city has two choices for cable TV either Acme TV or Beta TV. The transition matrix below shows the probability that customers switch companies at the end of the year. If a

4. Becca either walks to school or rides her bike. The transition matrix below shows the probability of her actions each day. What is the probability that Becca rides her bike two days

4.14 **Matrices Modeling Contexts Practice Set 2**

Name _____

Date _____ Period _____

Problems 1-3, determine which of the following matrices are regular.

1. $P = \begin{bmatrix} 0.2 & 0.8 \\ 0 & 1 \end{bmatrix}$

2. $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.15 & 0.85 \end{bmatrix}$

3. $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Problems 4-6, Suppose the transition matrix for a Regular Markov Chain. Find the stationary matrix S by solving $S \cdot T(\mathbf{x}) = S$

4. $T = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$

4.14 **Matrices Modeling Contexts Practice Set 3**

Name _____

Date _____ Period _____

Problems 1-2, find the stable distribution matrix for each scenario.

1. You raise money for a pet shelter each year as a charity contribution. It is shown that those who contribute one year, 40% will not contribute next year. Also, those who don't contribute this year, 20% will contribute next time. Find the final distribution matrix.

2. Two hardware stores compete with each other for market shares. 30% of the customers who shop at Burn's Hardware will switch to Dean's Tools every month. And 20% of the customers who shop at Dean's Tools will switch to Burn's Hardware every month.

$\begin{bmatrix} 0.8 & 0.1 & 0.1 \end{bmatrix}$

6. Becca either walks to school or rides her bike. The transition matrix below shows the probability of her actions each day. Assuming that the initial day is Monday, what is the probability matrix for Wednesday? What is the probability that Becca walked on Wednesday if she biked on Monday?

Initial Day:	Next Day: Walk	Next Day: Bike
Walk	1/2	1/2
Bike	1/4	3/4

7. A new public transit system has just begun to operate. In the first month of operation, a study found that 25% of commuters are using the new system, while 75% still travel in private vehicles. The transition matrix was determined from records of other transit systems.

A. What is the initial-state matrix?

B. What percentage of commuters will be using the new transit system after 1 month?

Current month:	Next month: Public	Next month: Private
Public	0.7	0.3
Private	0.2	0.8

Problems 7-9, Consider each scenario and solve.

7. Suppose the transition matrix for a tennis player is as follows, where **C** represent the cross-court shots and **D** represents shots down-the-line.

A. If the player hit the first shot cross-court, what is the probability they will hit the fourth shot cross-court?

B. What is the long-term shot distribution?

Current Shot	Next Shot: C	Next Shot: D
C	0.88	0.12
D	0.90	0.10

8. Hayden eats at his favorite Mexican restaurant every day at lunch. The transition matrix for his ordering behavior is shown below, assume that the initial day is Monday.

A. If Hayden orders a burrito on Monday, what is the probability matrix for Thursday?

Initial:	Next: Tacos	Next: Burrito
Tacos	0.5	0.5
Burrito	0.25	0.75

Problems 4-5, solve.

4. A transition matrix for students who switch majors each quarter is given below. The majors are Liberal Arts, Business and Science, denoted by A, B, and S respectively.

A. Find the probability of a Science major switching to a Business major during the first quarter.

Current Quarter:	Next Quarter: A	Next Quarter: B	Next Quarter: S
A	0.8	0.1	0.1
B	0.2	0.7	0.1
S	0.1	0.3	0.6

B. What is the probability of a Business major switching to Liberal Arts during their second semester?

C. Find the probability of a Science Major switching to Liberal Arts during their third quarter.

5. Three large corporations compete for a lead in the market share. The transition matrix for consumers switching from company to company each year is given below.

A. If the initial market share is 20% for Company I, 30% for Company II, and 50% for Company III, what will the market share be after the next year?

From:	To: I	To: II	To: III
I	0.6	02	0.2
II	0.2	0.5	0.3
III	0.3	0.3	0.4

Practice Sets

4.14 Matrices Modeling Contexts **Form A**
Set 1

Name _____

Date _____ Period _____

Consider a marketing campaign for cellphone covers where a start-up company initially has a 20% market share. The advertising department has a transition matrix shown below. Let A = Brand A covers and A' = a different brand of covers. Let's assume the advertising campaign is on a monthly basis.

$$\begin{array}{c} \\ A \\ A' \end{array}\begin{array}{c} \begin{array}{cc} A & A' \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$

1. What is the initial-state matrix in the form of $S_0 = [B \quad B']$?

2. Find the market share after 3 months.

3. If the probabilities remain valid over a long period of time, what happens to the company's market share for cell phone covers?

Grade:

4.14 Matrices Modeling Contexts **Form B**
Set 1

Name _____

Date _____ Period _____

Consider a marketing campaign for cellphone covers where a start-up company initially has a 22% market share. The advertising department has a transition matrix shown below. Let A = Brand A covers and A' = a different brand of covers. Let's assume the advertising campaign is on a monthly basis.

$$\begin{array}{c} \\ A \\ A' \end{array}\begin{array}{c} \begin{array}{cc} A & A' \end{array} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$

1. What is the initial-state matrix in the form of $S_0 = [B \quad B']$?

2. Find the market share after 3 months.

3. If the probabilities remain valid over a long period of time, what happens to the company's market share for cell phone covers?

Grade:

4.14 Matrices Modeling Contexts **Form A**
Set 2

Name _____

Date _____ Period _____

Suppose the transition matrix for a basketball player is as follows, where T represents a three-point shot and L represents a lay-up.

$$\begin{array}{c} \\ T \\ L \end{array}\begin{array}{c} \begin{array}{cc} T & L \end{array} \\ \begin{bmatrix} 0.6 & 0.4 \\ 0.8 & 0.2 \end{bmatrix} \end{array}$$

1. If the player hits the first lay-up, what is the probability they will hit the fourth lay-up?

2. If the player hits the first three-point shot, what is the probability they will hit the third three-point shot?

3. If the player hits the first three-point shot, what is the probability they will hit the fifth lay-up?

Grade:

4.14 Matrices Modeling Contexts **Form B**
Set 2

Name _____

Date _____ Period _____

Suppose the transition matrix for a basketball player is as follows, where T represents a three-point shot and L represents a lay-up.

$$\begin{array}{c} \\ T \\ L \end{array}\begin{array}{c} \begin{array}{cc} T & L \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.9 & 0.1 \end{bmatrix} \end{array}$$

1. If the player hits the first lay-up, what is the probability they will hit the fourth lay-up?

2. If the player hits the first three-point shot, what is the probability they will hit the third three-point shot?

3. If the player hits the first three-point shot, what is the probability they will hit the fifth lay-up?

Grade:

4.14 Matrices Modeling Contexts **Form A**
Set 3

Name _____

Date _____ Period _____

1. You raise money for a charity each year. It is shown that of those who contribute one year, 30% will not contribute the next year. Also, those who don't contribute this year, 25% will contribute next year. Find the final distribution matrix.

2. Two pizza places compete with each other for market share. 35% of the customers who each at Luigi's will switch to Al's every month. And 40% of Al's customers will switch to Luigi's each month. Find the final distribution matrix.

Grade:

4.14 Matrices Modeling Contexts **Form B**
Set 3

Name _____

Date _____ Period _____

1. You raise money for a charity each year. It is shown that of those who contribute one year, 20% will not contribute the next year. Also, those who don't contribute this year, 30% will contribute next year. Find the final distribution matrix.

2. Two pizza places compete with each other for market share. 25% of the customers who each at Luigi's will switch to Al's every month. And 20% of Al's customers will switch to Luigi's each month. Find the final distribution matrix.

Grade:

Daily Lesson Quiz