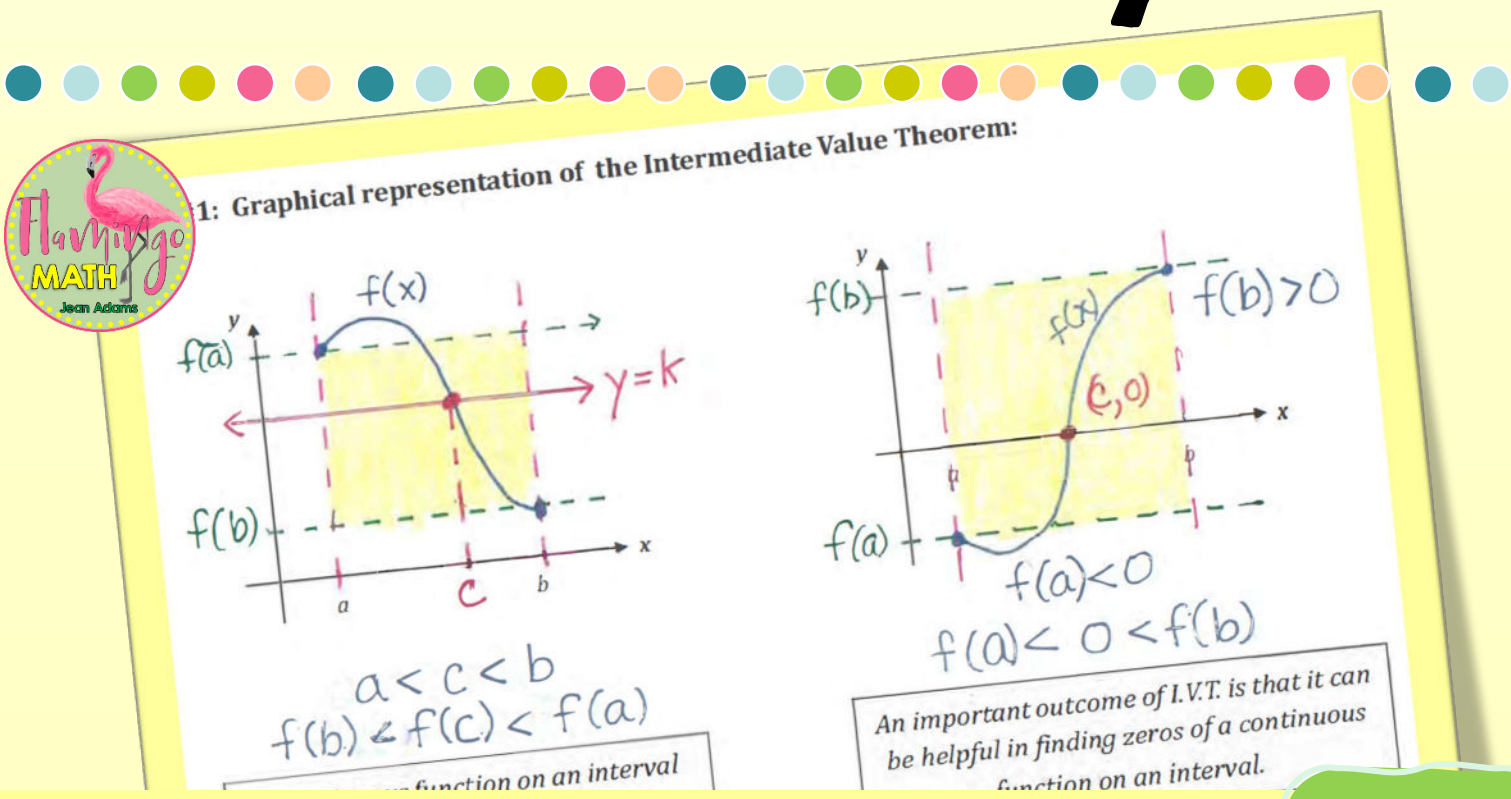
AP® Calculus

Limits and Continuity



Guided Notes 1

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AP CALCULUS CURRICULUM

Scope and Sequence

Unit 1: Limits and Continuity

Lesson	Skills & Learning Objectives
[1.1] Concept of Instantaneous Rate of Change	❑ Interpret rate of change at an instant in terms of average rates of change
[1.2] Understanding Limits Graphically and Numerically	❑ Evaluate the limit, if it exists by graphical and numerical methods ❑ Find one-sided limits and general limits
[1.3] Properties of Limits	❑ Use properties of limits to evaluate graphically, numerically, and algebraically
[1.4] Limits of Composite Functions	❑ Evaluate the limit of composite functions, if it exists by graphical and numerical methods. ❑ Find one-sided limits and general limits of composite functions
[1.5] Finding Limits by Analytic Methods	❑ Evaluate limits analytically, including substitution, cancellation, rationalization, and special trig rules
[1.6] Limits of Transcendental Functions	❑ Evaluate limits of exponential and trigonometric functions
[1.7] Limits and Continuity	❑ Apply the definition of continuity to determine whether a function is continuous at a point
[1.8] Infinite Limits and Limits at Infinity	❑ Compare and contrast infinite limits and limits at infinity ❑ Use properties of infinite limits to find asymptotes and describe function behavior. ❑ Use relative rates of growth of power, logarithmic, and exponential functions to analyze limits
[1.9] Intermediate Value Theorem	❑ Understand and apply the Intermediate Value Theorem

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Lesson I: The Concept of Instantaneous Rate of Change

Topic 1.1: Introducing Calculus: Can Change Occur at an Instant?

As students of life, and algebra, you have studied many rates of change for different situations: how fast we drive, how our savings grow over time, the rate that the world population changes, our growth rate every year, unemployment rates, and so much more. In each of these studies, we can express one variable in terms of another, that is, as a function, $y = f(x)$.

Velocity

Consider an object that travels in a straight line. The **average velocity** over some given time interval can be defined as the ratio of the change in position (**displacement** or **net change)** to the elapsed time period:

Average Velocity =

However, when we think of velocity, we usually mean **instantaneous velocity**, which indicates the speed and direction of some object at a particular moment, which could be faster or slower than the average velocity.

Can you explain why we can't define instantaneous velocity as a ratio?

How can you estimate an instantaneous velocity?

Let's generalize that big idea with this principle:

Average velocity over a very small time interval is very close to instantaneous velocity.

EX #1: If you climb to the top of the Tower of Pisa and drop a coin to the ground, estimate the instantaneous velocity at $t = 0.7$ seconds. Use Galileo's formula $s(t) = 16t^2$ to compute the average velocity, in feet/second , over the time intervals listed in the table below.

Time Interval	Average Velocity
$[0.7, 0.71]$	22.56
$[0.7, 0.705]$	
$[0.7, 0.7001]$	
$[0.7, 0.70005]$	
$[0.7, 0.700001]$	

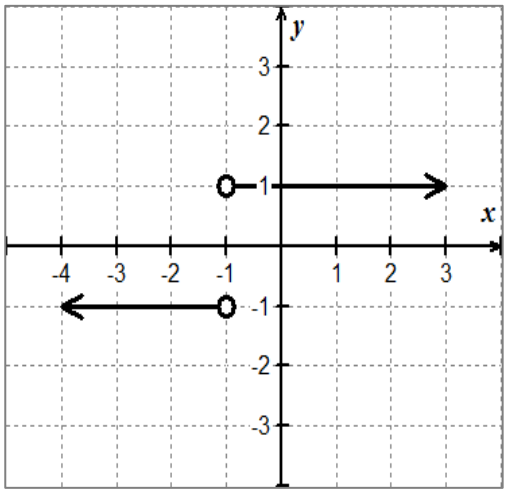
What is your estimate for the ***Instantaneous Rate of Change*** (IROC) at $t = 0.7$ second?



Photo: Jean Adams

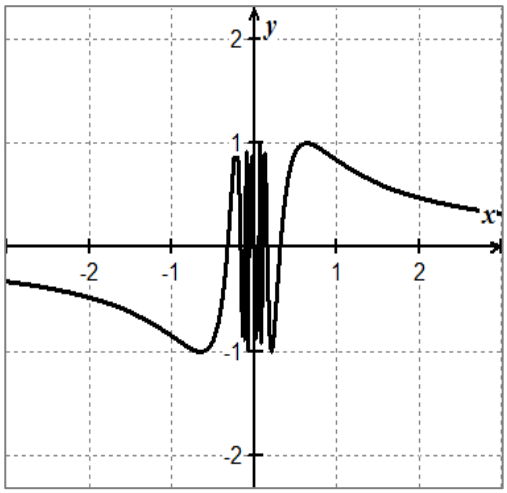
EX #4: Limits can fail to exist in three situations:

CASE 1: _____



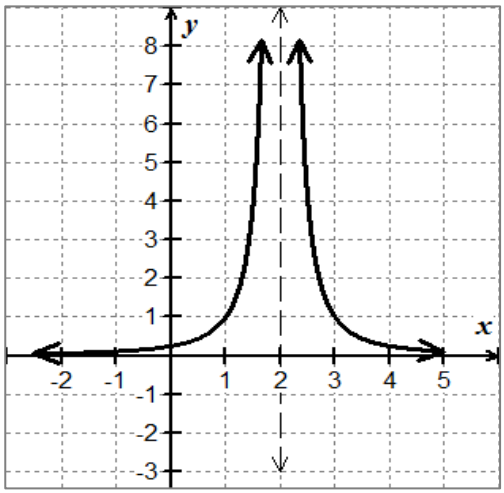
Justify why the limit does not exist at $x = -1$ for $f(x) = \frac{|x+1|}{x+1}$

CASE 2: _____



Justify why the limit does not exist at $x = 0$ for $f(x) = \sin\left(\frac{1}{x}\right)$

CASE 3: _____



Justify why the limit does not exist at $x = 2$ for $f(x) = \frac{1}{(x-2)^2}$

Lesson 3: Properties of Limits

Topic 1.5: Determining Limits Using Algebraic Properties of Limits

There will be times when you need to consider limits of functions symbolically, such as, $f(x)$, $g(x)$, and $h(x)$. When you are given combinations, compositions, and other modifications of these nonspecific functions, there are some important properties of limits that will prove essential to these general function types.

LIMIT PROPERTIES

Let b and c be real numbers, let n be a positive integer, and let f and g be functions where $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exists, then

1. Constant Rule:

$$\lim_{x\to c} k = k$$

2. Identity Rule:

$$\lim_{x\to c} x = c$$

3. Coefficient Rule:

$$\lim_{x\to c}[b \cdot f(x)] = b \cdot \lim_{x\to c} f(x)$$

4. Sum or Difference Rule:

$$\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$$

5. Product Rule:

$$\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

6. Quotient Rule:

$$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \; ; \lim_{x\to c} g(x) \neq 0$$

7. Power Rule:

$$\lim_{x\to c}[f(x)]^n = \left(\lim_{x\to c} f(x)\right)^n$$, n is rational

8. Composite Function Rule:

$$\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right)$$

9. Root Rule:

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$$ if the root on the right side exists.

Being skilled at reciting these rules in English will help you truly understand them. For example, The Sum or Difference Rule can be stated as "*The limit of a sum (difference) is the sum (difference) of the limits.* Can you give a description for each property in your own words?

Lesson 6: Limits of Transcendental Functions

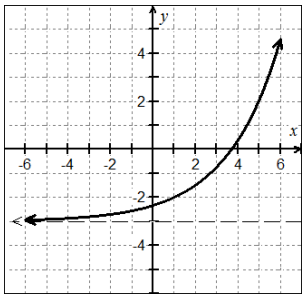
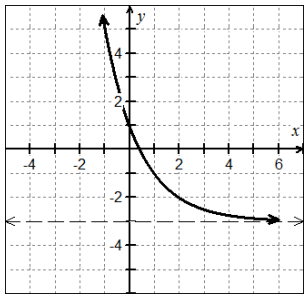
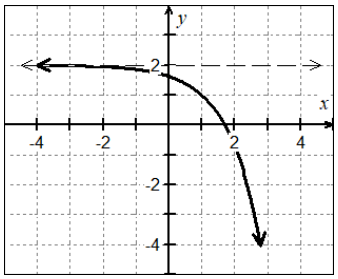
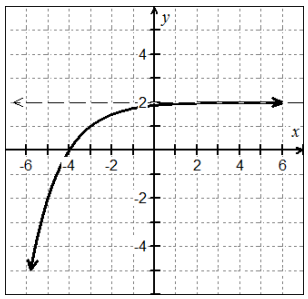
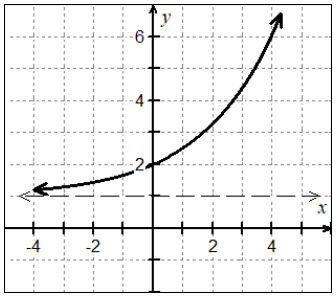
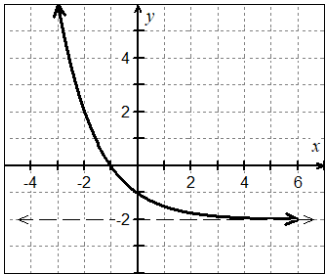
If you research the definition of a **transcendental function** you will find this explanation on Wolfram's Math World:

> *A function which is not an algebraic function. In other words, a function which "transcends," i.e., cannot be expressed in terms of, algebra. Examples of transcendental functions include the exponential function, the trigonometric functions, and the inverse functions of both.*

Topic 1.9: Connecting Multiple Representations of Limits

Analyzing Limits of Exponential Functions

EX #1: Recall that exponential equations are written in the form $y = ab^{(x-h)} + k$. You will need to find limits of exponential functions without the aid of a graph or calculator in this course. Do you remember the rules for transformations of exponential functions? Evaluate the limits using the graphs and look for patterns.

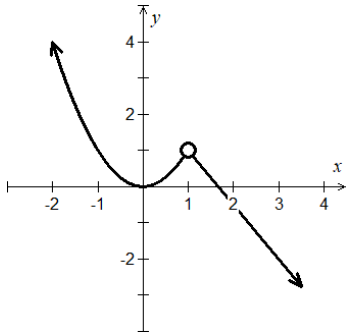
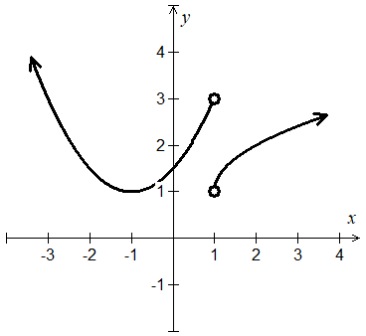
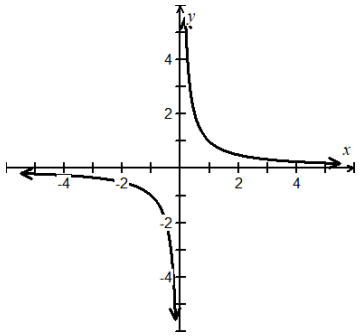
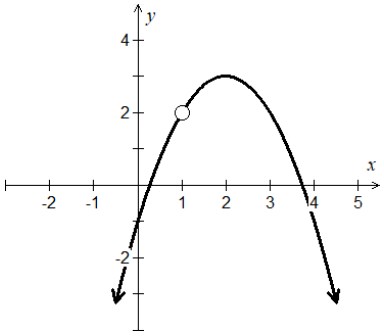
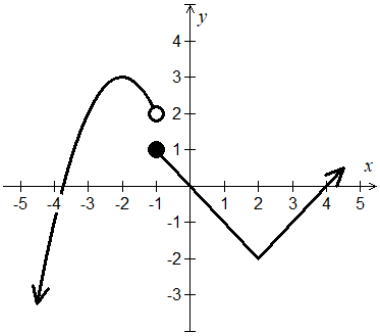
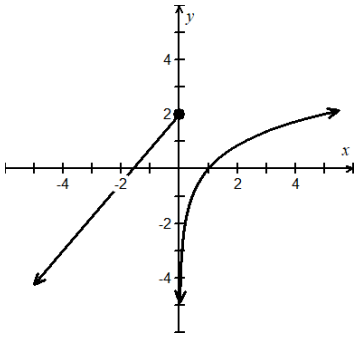
Exponential Growth	Exponential Decay
A. $f(x) = \left(\frac{3}{2}\right)^{x-1} - 3$ $\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} f(x) =$ 	**D.** $f(x) = \left(\frac{1}{2}\right)^{x-2} - 3$ $\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} f(x) =$ 
B. $f(x) = -e^{x-1} + 2$ $\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} f(x) =$ 	**E.** $f(x) = -\left(\frac{1}{2}\right)^{x+3} + 2$ $\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} f(x) =$ 
C. $f(x) = \left(\frac{2}{3}\right)^{-x} + 1$ $\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} f(x) =$ 	**F.** $f(x) = 2^{-x} - 2$ $\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} f(x) =$ 

*While **limits** told us where a function intended to go.*

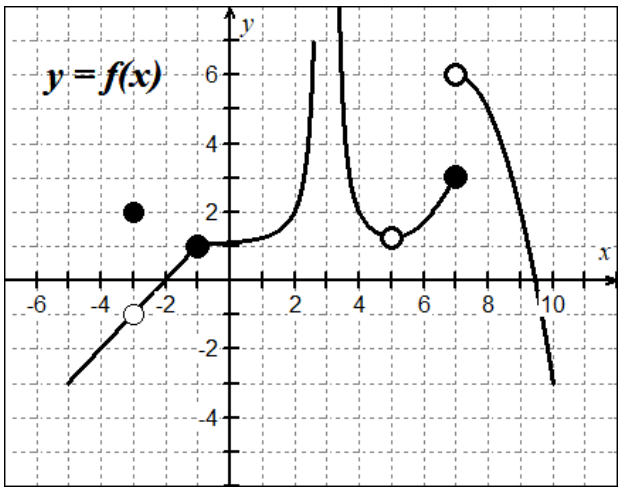
***Continuity** guarantees that the function actually made it there.*

Classifying Discontinuities

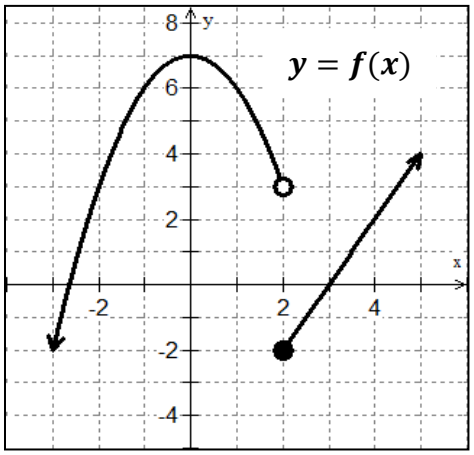
Topic 1.10: Exploring Types of Discontinuities

Removable or Point (Holes) **2-sided limit exists**	**Non-Removable**	
	Jump 1-sided limits exists	**Infinite** At least one of the 1-sided limits doesn't exist
		
		

EX #2: Find the points (intervals) at which the function is continuous, and the points at which the function is discontinuous on the interval $-5 < x < 10$.

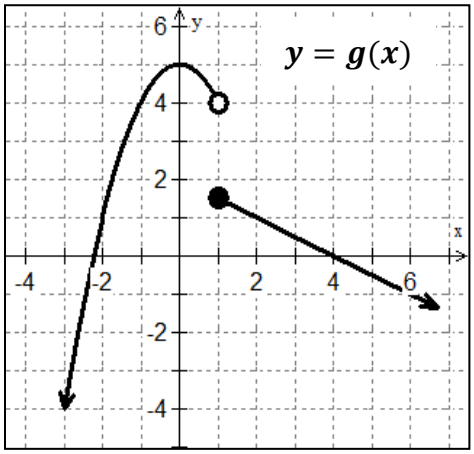


EX #2: Does the Intermediate Value Theorem apply on the specified intervals for the piecewise functions shown below? Be sure to explain completely.



A. Is there a value of $x = c$ on the interval $[-2, 5]$ such that $f(c) = 1$?

B. Does I.V.T. guarantee a value of c on the interval $[-2, 5]$ such that $f(c) = 1$? Justify.



C. Is there a value of $x = c$ on the interval $[1, 6]$ such that $g(c) = 3$?

D. Does I.V.T. guarantee a value of c on the interval $[1, 6]$ such that $g(c) = 3$? Justify.

What three conditions are necessary to apply the Intermediate Value Theorem?

1.

2.

3.

EX #6: Let $g(x)$ be a continuous function. Selected values of $y = g(x)$ are given in the table below. For which value of k will the equation $g(x) = 3/4$ have **at least two solutions** on the closed interval $[2, 8]$?

x	2	3	4	5	8
$g(x)$	3	4	k	5	2

(A) 1 **(B)** $\frac{3}{4}$ **(C)** $\frac{9}{16}$ **(D)** $\frac{3}{2}$

EX #7: For the function $f(x) = \begin{cases} (x-3)^2, x = 5 \\ 6, \ 5 < x \leq 10 \end{cases}$. Find $f(5)$ and $f(10)$. Does IVT guarantee a y-value k on $5 \leq x \leq 10$ such that $f(5) < f(k) < f(10)$? Justify your answer.

EX #8: The functions f and g are continuous for all real numbers. The table below gives values of the functions at selected values of x The function h is given by $h(x) = g\big(f(x)\big) + 2$**.** Explain why there must be a value w for $1 < w < 6$ such that $h(w) = 0$.

x	1	2	6	8
$f(x)$	2	9	8	13
$g(x)$	3	-12	5	28