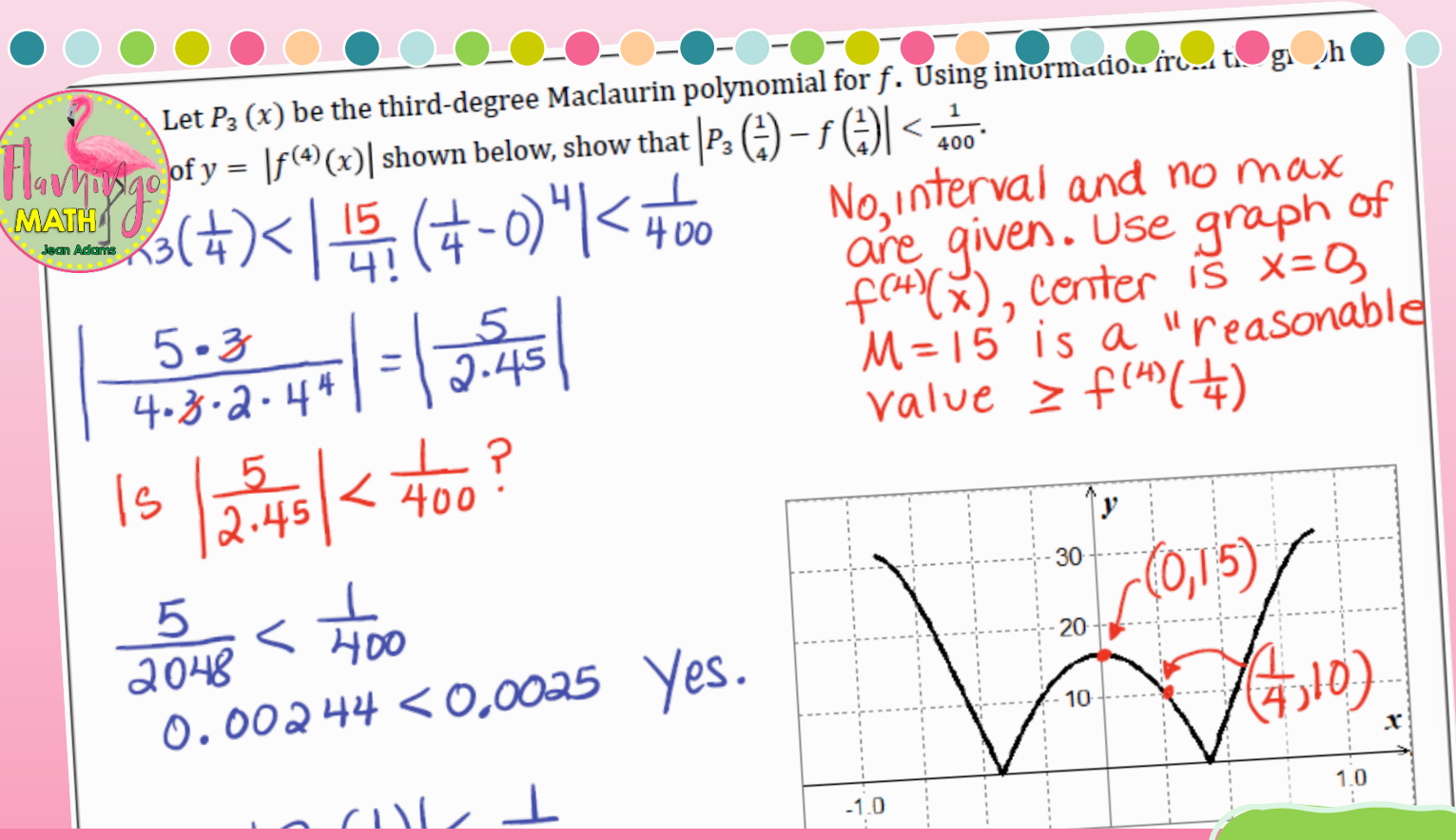
AP® Calculus BC

Lagrange Error Bound & Accuracy



Completely Editable

10.7

Lesson 10.7: Lagrange Error Bound and Accuracy

Topic 10.12: Lagrange Error Bound

At this point in our study of Calculus, we have explored many methods to approximate values for a function. We could simply estimate by looking at a graph, use tangent line approximations or Euler's method, and now Taylor polynomials.

Remainder, Error and Accuracy of a Taylor Polynomial

When we use Taylor polynomial approximations it would be a good idea to know how accurate they are to the exact values of an infinite series. The function value by the Taylor polynomial can be defined by the **concept of a remainder** defined as

$$f(x) = P_n(x) + R_n(x)$$

Exact value = Approximate value + Remainder

So, the absolute value of the remainder $R_n(x)$ is called the error.

$$\text{Error} = |R_n(x)| = |f(x) - P_n(x)|$$

Graphically

When we use a Taylor polynomial $T_n(x)$, centered at $x = c$ to approximate a function, $f(x)$ at a value $x = a$ near the center, we can say the following:

If: Function = Polynomial approximation + Remainder

Then: Remainder = Function – Polynomial app

In order to use Taylor polynomials effectively, we
of the error. The remainder given in the theorem
or the **Lagrange error bound**. This gives us a **bo**

Taylor's

If a function f is differentiable through order n
the interval, there exists a number a between x

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c$$

$$R_n(x) = \frac{f^{(n+1}}{(n+}$$

Instead of finding the exact error of our approximations, ***we will use two rules that will help us find an upper bound for the error***. In this way, we will know the most that our approximation is off the actual mark.

1. If the series is an Alternating Series, the error will be no greater than the absolute value of the first unused term (remember this rule from the previous unit).
2. If a non alternating series is approximated, we will use Taylor's Inequality Theorem (Lagrange Form of the Remainder) stated below.

Taylor's Inequality (Lagrange Form of the Remainder)

$|R_n(x)| \le \left|\frac{M}{(n+1)!}|x-c|^{n+1}\right|$ where c is the center.

M (for maximum) is defined to be the maximum value of $f^{n+1}(a)$ where $x \le a \le c$.
Bounds on $R_n(x)$ found using this form are also called Lagrange error bounds.

For $n = 0$, Taylor's Theorem states that if f is differentiable in an interval containing c, then for each x in the interval, there exists a between x and c such that

$$f(x) = f(c) + f'(a)(x-c) \text{ or } f'(a) = \frac{f(x)-f(c)}{x-c}$$

Do you recognize this as the Mean Value Theorem?

Vocabulary Check for Understanding

1. _____ of _____ on
_____. We can find M by substituting _____
_____: _____.
2. _____,
_____ will tell us _____.
3. _____ween the center, c, and the value $x = a$, is not given or
_____
4. _____num or the interval, assume the interval is from
_____.
5. _____f we increase n by adding more terms to the polynomial,
_____ in size. Second, as we move away from the center of the
_____.

EX #1: The function f has continuous derivatives for all real numbers x. Assume that $f(3) = 5$, $f'(3) = 2, f''(3) = -4, f'''(3) = 7$.

A. Write a third-degree Taylor polynomial for f about $x = 3$ and use it to approximate $f(3.2)$. Give three decimal places.

B. The fourth derivative of f satisfies the inequality $\left|f^{(4)}(a)\right| \le 35$ for all x in the interval $[3, 3.2]$. Use this information to find a bound for the error found in (A).

C. Find an interval $[a, b]$ such that $a \le f(3.2) \le b$. Give three decimal places.

D. Can $f(3.2)$ equal 5.331? Explain why or why not.

E. Can $f(3.2)$ equal 5.334? Explain why or why not.

EX #2: Estimate $\sin(0.5)$ using a Maclaurin polynomial. Wh
that assures an error smaller than 0.001?

EX #3: Let $P_3(x)$ be the third-degree Maclaurin p
of $y = \left|f^{(4)}(x)\right|$ shown below, show that

Errors and Accuracy for Alternating Series

Topic 10.10: Alternating Series Error Bound

NOTE: If an error calculation is required on the AP exam, all work must be shown so the calculator can only be used as a method to check your computations.

EX #4: Use the Maclaurin series for $y = e^x$ to estimate $\frac{1}{\sqrt{e}}$ accurate to four decimal places.

EX #5: Using the Maclaurin series for $\cos x$.

A. Write the fourth-degree polynomial and use it to compute $\cos\left(\frac{\pi}{12}\right)$.

B. Use the Lagrange error bound to show that $\left|f\left(\frac{\pi}{12}\right) - P_4\left(\frac{\pi}{12}\right)\right| \le \frac{1}{5000}$ for $0 \le x \le 1$.

C. Use the fourth-degree Taylor polynomial for $\cos x$ about $x = 0$ to find an approximation for $\cos 0.8$ and determine the accuracy of the approximation.

NOTES

10.7 **Lagrange Error Bound and Accuracy Homework**

Name _____

Date _____ Period _____

Problems 1 - 8, Use Taylor's Inequality to determine the error bounds of the approximations.

1. $e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$

2. $\cos(0.4) \approx 1 - \frac{(0.4)^2}{2!} + \frac{(0.4)^4}{4!}$

3. Estimate the value of $e^{1.5}$ by using a Taylor polynomial centered at $x = 2$, find the least degree of the polynomial needed to have an error less that 0.001.

4. Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 1$ is given by $P(x) = 6 - 9(x-1)^2 + 4(x-1)^3$. Suppose $\left|f^{(4)}(x)\right| \leq 6$ for all x on the interval $[0,1]$. Use the Lagrange error bound to justify why $f(0)$ is negative.

5. Find a 4th degree Taylor polynomial for $\ln x$ centered at $x = 4$.

A. Use the polynomial to approximate $\ln(4.2)$.

B. Find the Lagrange error bound on the interval $[4, 4.5]$.

6. Given $f(x) = \sqrt{x}$

A. Write a 2nd degree Taylor polynomial for $f(x)$ about $x = 4$.

B. Use the polynomial to approximate $f(4.1)$.

C. Find the Lagrange error bound for the approximation on the interval; $[4, 4.1]$

7. Let $f(x)$ be a function that is continuous and differentiable at all real numbers, and $f(2) = 1$, $f'(2) = 5$, $f''(2) = 7$, and $f'''(2) = -3$.

A. Write a 3rd order Taylor polynomial for $f(x)$ about $x = 2$.

B. Use the polynomial to approximate $f(1.9)$.

C. If $f^{(4)}(x) \leq 4$ for all x, find the Lagrange error bound for the approximation.

8. Let $f(x)$ be a function that is continuous and differentiable at all real numbers. Assume $f(4) = 5$, $f'(4) = 7$, $f''(4) = 18$, $f'''(4) = 24$. Also, $\left|f^{(4)}(x)\right| \leq 45$ for all x in the interval $[4, 4.2]$.

A. Find a 3rd degree Taylor polynomial about $x = 4$ to estimate $f(4.2)$.

B. Find the maximum possible error for $f(4.2)$.

D. Could $f(4.2) = 6.785$? Explain why or why not.

Practice Sets

10.7 Lagrange Error Bound **Form A**

Name _____

Date _____Period _____

Let $f(x)$ be a function that is continuous and differentiable at all real numbers, and $f(3) = 5$, $f'(3) = -2$, $f''(3) = 4$ and $f'''(3) = -1$. Also, $\left|f^{(4)}(x)\right| \leq 6$ for all x in the interval $[3, 3.2]$.

1. Write a 3rd order Taylor polynomial for $f(x)$ about $x = 3$.

2. Use your polynomial to approximate $f(3.2)$.

3. Find the maximum possible error for $f(3.2)$.

4. Find an interval $[a, b]$ where $a \leq f(3.2) \leq b$.

10.7 Lagrange Error Bound **Form B**

Name _____

Date _____Period _____

Let $f(x)$ be a function that is continuous and differentiable at all real numbers, and $f(2) = 7$, $f'(2) = -4$, $f''(2) = 6$ and $f'''(2) = -2$. Also, $\left|f^{(4)}(x)\right| \leq 9$ for all x in the interval $[2, 2.1]$.

1. Write a 3rd order Taylor polynomial for $f(x)$ about $x = 2$.

2. Use your polynomial to approximate $f(2.1)$.

3. Find the maximum possible error for $f(2.1)$.

4. Find an interval $[a, b]$ where $a \leq f(2.1) \leq b$.

Daily Lesson Quiz