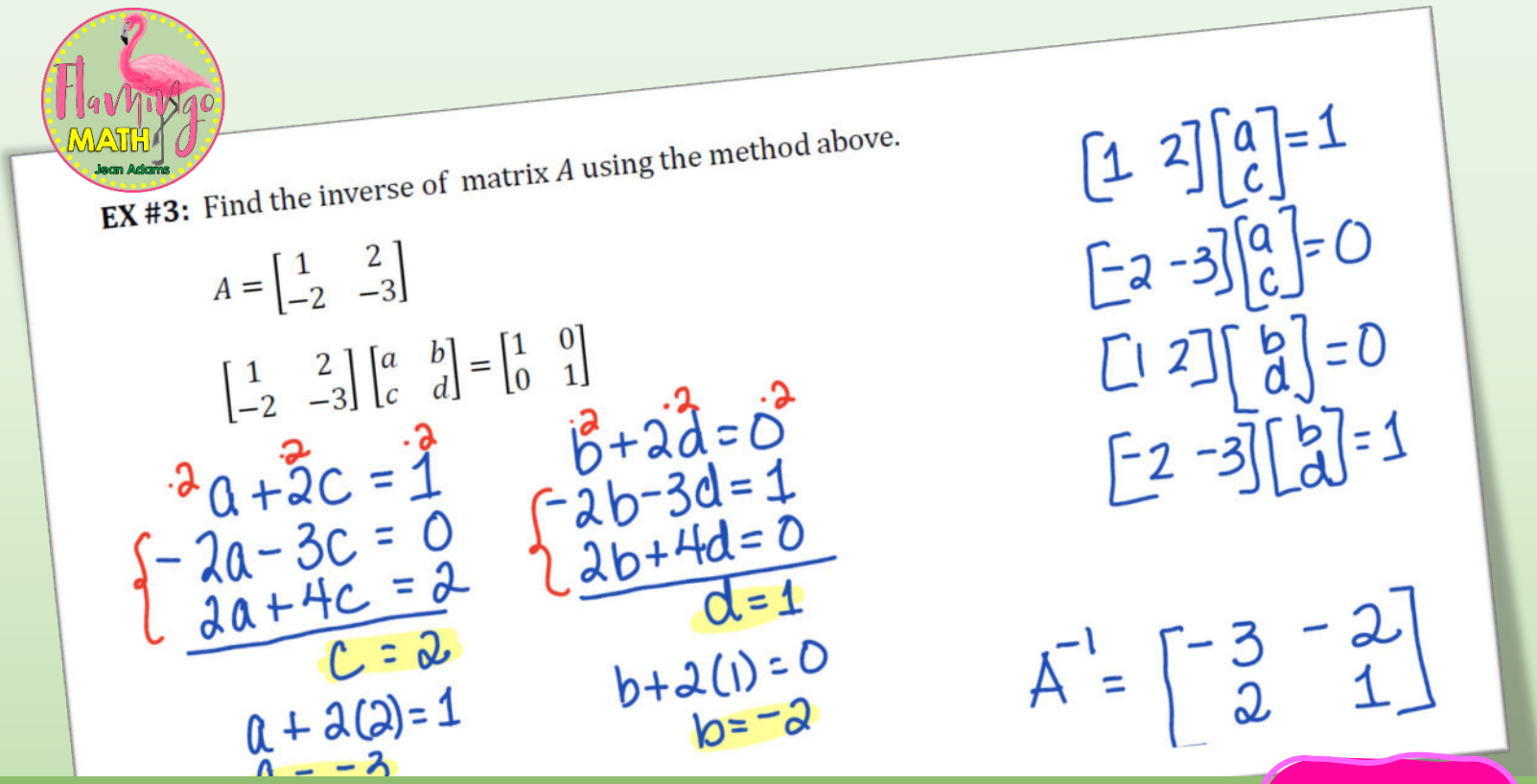
AP® Precalculus

The Inverse and Determinant of a Matrix



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4.11

Lesson II-A: The Inverse and Determinant of a Matrix

Consider the real number equation $ax = b$. To solve this equation for x, multiply each side of the equation by the multiplicative inverse of a, that is, $a^{-1}(a \neq 0)$. This general idea is similar in matrix operations. The product of **matrix A** and its inverse A^{-1} equals the **identity matrix.**

Definition of the Identity Matrix

Any $m \times n$ matrix with the same number of rows as columns is known as a **square matrix**. The identity matrix, I, is a ***square matrix*** where the diagonal entries are 1's and all other entries are 0's.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition of the Inverse of a Square Matrix

Let A be a square $n \times n$ matrix. If there exists an $n \times n$ matrix, A^{-1}, read "A inverse," where $AA^{-1} = A^{-1}A = I$, then A^{-1} is called the inverse of matrix A.
The **identity matrix** acts as a 1 in matrix algebra.

It should be noted that it is not always true that $AB = BA$. If A ad B are both square matrices and $AB = I$, it can be shown that $BA = I$.

EX #1: Show that B is the inverse of A by showing $AB = I$ and $BA = I$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

Finding Inverse Matrices

A matrix A is called **invertible** (or **nonsingular**) if the matrix has an inverse, otherwise A is called **singular**. *A non-square matrix cannot have an inverse.* Not all square matrices have inverses, but if a matrix does have an inverse, that inverse is unique.

Also consider, if matrix A is of order $m \times n$ and matrix B is of order $n \times m$ where $m \neq n$, the products AB and BA are different orders and cannot be equal to each other.

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Row Operations

Row Operation – on an augmented matrix any of the following:
switch two rows, multiply a row by a constant, add one row to another.

One method to find the multiplicative inverse is by augmenting the matrix with the **identity** (shown at right). Then, you can perform row operations:

$$\left[\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

1. Switch any two rows.
2. Multiply a row by a constant.
3. Add one row to another row.
4. Combine any of these operations.

Finding an Inverse Matrix

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \vdots I]$.
2. If possible, use elementary row operations on the entire matrix. This result will be the matrix $[I \vdots A^{-1}]$. If this isn't possible, then A is not invertible.
3. Check your result that $AA^{-1} = I = A^{-1}A$

EX #2: Find the inverse of matrix A using the method above.

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$$

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Finding an Inverse Matrix

A second method to finding an inverse matrix is based on the idea of a matrix equation.

$AX = I$. Let A be an invertible matrix, and I is the identity matrix. Matrix X is known as the variable matrix, such that for a 2×2 matrix:

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Using matrix multiplication, it is possible to solve two systems of equations and find the values of the inverse matrix which will give the identity.

EX #3: Find the inverse of matrix A using the method above.

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

EX #4: Use one of the methods to show that matrix B is not invertible.

$$B = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

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Find an Inverse Matrix with a TI-Nspire Calculator

EX #5: Find the inverse of matrix A.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

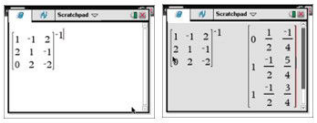
With the TI-Inspire, pick the template and set the matrix size:



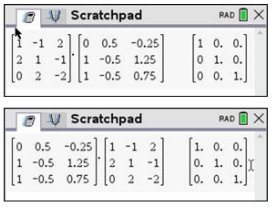
Enter the elements in each row and column:



Calculate the inverse:



Now verify that $AA^{-1} = A^{-1}A = I$. Explain.



NOTE: If you receive "error Singular Matrix" this indicates there is no inverse possible.

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Guided Notes

Find an Inverse Matrix with a TI-84 Plus CE Calculator

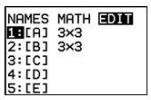
EX #5: Find the inverse of matrix A.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

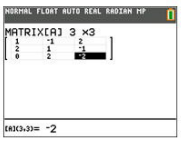
With the TI-84 Plus CE:

2ND X^{-1} EDIT

1. Press 1



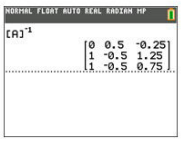
Enter the elements in each row and column:



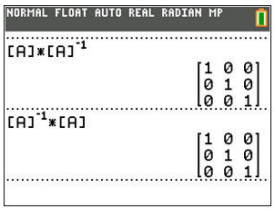
Calculate the inverse:

2ND X^{-1} 1

X^{-1} ENTER



Now verify that $AA^{-1} = A^{-1}A = I$. Explain.



NOTE: If you receive "error Singular Matrix" this indicates there is no inverse possible.

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Lesson II-B: The Inverse and Determinant of a Matrix

When you need to find the inverse of a 2 × 2 matrix, you can use a special formula instead of using the augmented matrix or matrix multiplication as we saw in the last lesson. This formula uses the **determinant of the matrix.**

The Determinant of a 2 x 2 Matrix

The determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

This value can be calculated with or without technology and is denoted $\det(A)$.

Inverse of a 2 x 2 Matrix

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $ad - bc \neq 0$, then the inverse matrix

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

EX #6: Find the determinant of each of the following matrices.

A. $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$ **B.** $B = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$ **C.** $C = \begin{bmatrix} 0 & 2 \\ 4 & 3 \end{bmatrix}$

D. Will all of the matrices above have an inverse? Explain.

EX #7: Find the inverse of the matrix B.

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

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Determinants as Area

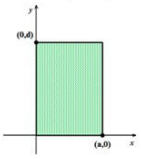
In this final application, we will look at the geometric interpretation of determinants as area. Since our course does not include finding the determinant of a 3 × 3 matrix, the concept of volume will be omitted.

Area of a Parallelogram

If A is a 2 × 2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

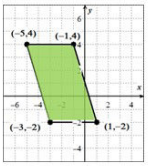
If A is a 3 × 3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

$$\left\{\begin{matrix}\textbf{area of} \\ \textbf{rectangle}\end{matrix}\right\} = \left|\det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right| = |ad|$$



*** Note: If the $\det(A) = 0$, then the vectors are parallel.

EX #8: Calculate the area of the parallelogram determined by the points $(-5,4)(-1,4)(-3,-2)$ and $(1,-2)$.



A. Translate the parallelogram so that the origin is a vertex.

B. Create matrix A from the vertices that are adjacent to the origin.
Name the translated image using $A = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

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Guided Notes

4.11 **The Inverse and Determinant of a Matrix Practice Set 1**

Name _____

Date _____ Period _____

Problems 1-2, show that B is the inverse of A.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$

Problems 3-8, find the inverse of the matrix, if it exists using the $A \cdot I = A^{-1}$ method.

3. $\begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & 7 \\ 2 & -3 & -1 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -2 \\ 5 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & -2 \\ 5 & -2 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

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Problems 9-10, use the matrix feature of a graphing utility to find the inverse of the matrix, if possible.

9. $\begin{bmatrix} 1.24 & 2.15 \\ -4.1 & -5.2 \end{bmatrix}$

10. $\begin{bmatrix} \frac{3}{8} & -\frac{3}{2} \\ \frac{9}{20} & \frac{2}{5} \end{bmatrix}$

Problems 11-14, find the inverse using the method $A \cdot X = I$.

11. $\begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 9 & -24 \\ 12 & 32 \end{bmatrix}$

13. $A = \begin{bmatrix} -4 & 0 \\ 1 & 3 \end{bmatrix}$

14. $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

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4.11 **The Inverse and Determinant of a Matrix Practice Set 2**

Name _____

Date _____ Period _____

Problems 1- 4, find the determinant of the matrix.

1. $[7]$

2. $[-12]$

3. $\begin{bmatrix} 6 & -2 \\ 9 & -3 \end{bmatrix}$

4. $\begin{bmatrix} -5 & 3 \\ 6 & 2 \end{bmatrix}$

Problems 5 - 8, use the formula for a 2×2 matrix to find the inverse, if possible.

5. $\begin{bmatrix} 3 & -3 \\ 4 & -8 \end{bmatrix}$

6. $\begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 10 & 2 \\ -5 & -1 \end{bmatrix}$

Problems 9 - 12, solve for x.

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Problems 13-14, find the value of the constant k such that $B = A^{-1}$.

13. $A = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}$

14. $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} k & \frac{1}{7} \\ -\frac{1}{7} & \frac{3}{7} \end{bmatrix}$

Problems 15-16, find the area of the parallelogram whose vertices are listed.

15. $(0,0), (0,4), (5,8), (5,3)$

16. $(-1,5), (2,5), (0,0), (-3,0)$

Problems 17-18, two adjacent he sides of a parallelogram are formed by vectors,

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Practice Sets

4.11 The Inverse and Determinant of a Matrix Set 1 **Form A**

Name _____

Date _____ Period _____

1. Find the inverse of the matrix, if it exists using the $A \cdot I = A^{-1}$ method.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

2. Use a graphing utility to find the inverse of the matrix, if possible.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{3}{5} \\ -\frac{1}{3} & \frac{2}{5} \end{bmatrix}$$

3. Find the inverse using the $A \cdot X = I$ method.

$$A = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix}$$

Grade:

4.11 The Inverse and Determinant of a Matrix Set 1 **Form B**

Name _____

Date _____ Period _____

1. Find the inverse of the matrix, if it exists using the $A \cdot I = A^{-1}$ method.

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

2. Use a graphing utility to find the inverse of the matrix, if possible.

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{5} \\ -\frac{1}{2} & \frac{3}{5} \end{bmatrix}$$

4.11 The Inverse and Determinant of a Matrix Set 2 **Form A**

Name _____

Date _____ Period _____

1. Find the determinant of the matrix.

$$A = \begin{bmatrix} -3 & 2 \\ 5 & -7 \end{bmatrix}$$

2. Use the formula for a 2×2 matrix to find the inverse, if possible.

$$A = \begin{bmatrix} 5 & -2 \\ 1 & -3 \end{bmatrix}$$

3. Solve for x.

$$\begin{vmatrix} x+2 & 2 \\ 3 & x \end{vmatrix} = 9$$

4. Find the area of the parallelogram whose vertices are $(0,0), (0,5), (4,1), (4,6)$

Grade:

4.11 The Inverse and Determinant of a Matrix Set 2 **Form B**

Name _____

Date _____ Period _____

1. Find the determinant of the matrix.

$$A = \begin{bmatrix} -2 & 5 \\ 3 & -6 \end{bmatrix}$$

2. Use the formula for a 2×2 matrix to find the inverse, if possible.

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

3. Solve for x.

$$\begin{vmatrix} x+3 & 3 \\ 2 & x \end{vmatrix} = -2$$

4. Find the area of the parallelogram whose vertices are $(0,0), (0,6), (3,4), (3,10)$

Grade:

Daily Lesson Quiz