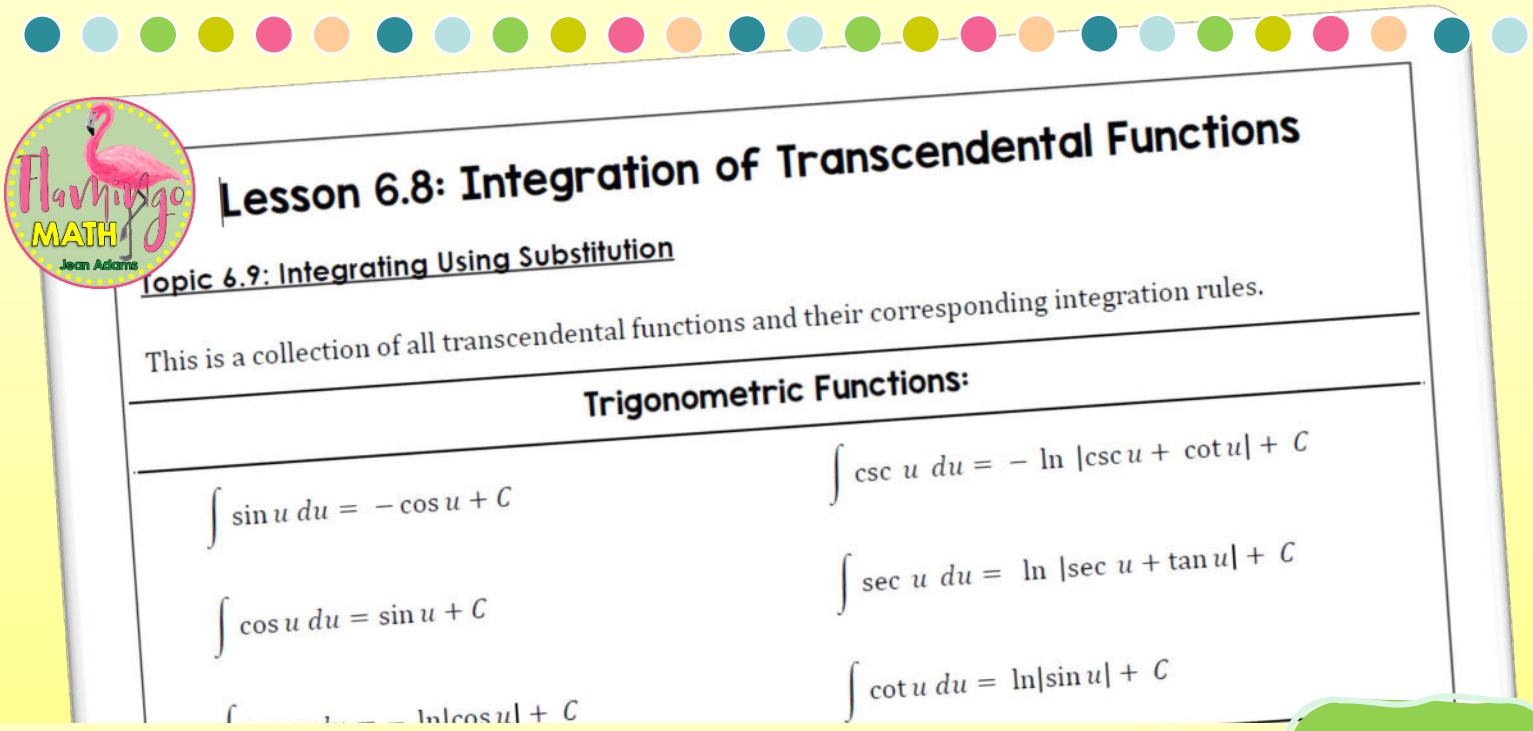
AP® Calculus-BC

Integration

Transcendental Functions

Calculus BC Version



Lesson 6.8: Integration of Transcendental Functions

Topic 6.9: Integrating Using Substitution

This is a collection of all transcendental functions and their corresponding integration rules.

Trigonometric Functions:

$$\int \sin u \, du = -\cos u + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \ln|\cos u| + C$$

Completely Editable 6.8

Lesson 6.8-BC: Integration of Transcendental Functions

Topic 6.9: Integrating Using Substitution

Integration of trigonometric functions come in all different formats. In this lesson, we will use trig identities and rules for integrals involving products of trig functions to integrate combinations of these functions.
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$\int \cos u \, du = \sin u + C$

$\int \sec u \, du = \ln|\sec u + \tan u| + C$

$\int \tan u \, du = -\ln|\cos u| + C$

$\int \cot u \, du = \ln|\sin u| + C$

Logarithmic and Exponential Functions:

Let u be a differentiable function of x.

$\int e^u \, du = e^u + C$

$\int \frac{du}{u} = \ln|u| + C$

$\int a^u \, du = \left(\frac{a^u}{\ln a}\right) + C$

EX #1: Find the following:

A. $\int \frac{1}{x \ln x^3} \, dx$

B. $\int \frac{\sin\theta}{2 - 3\cos\theta} \, d\theta$

C. $\int x^2 e^{\frac{x^3}{2}} \, dx$

D. $\int 2^{\sin x} \cos x \, dx$

EX #2: Find the definite integral:

A. $\int_0^{\pi/2} \frac{1 - \sin x}{x + \cos x} \, dx$

B. $\int_1^3 \frac{e^{\frac{3}{x}}}{x^2} \, dx$

C. $\int_0^3 \frac{2e^{2x}}{1 + e^{2x}} \, dx$

D. $\int_0^1 x \, 5^{x^2} \, dx$

Let's explore a few of the more challenging topics of integration that will appear in the BC curriculum. We will start with inverse trig functions and move on to products of powers.

Integrals of Inverse Trigonometric Functions

$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$	$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$
$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$	$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
$\int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1} x + C$	$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{\lvert u\rvert}{a}\right) + C$

EX #3: Evaluate the following.

A. $\int \frac{1}{x^2+9} \, dx$

B. $\int \frac{dx}{5 + 4x^2}$

C. $\int \frac{4x}{\sqrt{9-4x^4}} \, dx$

D. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} \, dx$

Integration by Completing the Square

Topic 6.10: Integrating Functions Using Long Division and Completing the Square

We will also see another level of difficulty added to integration problems using inverse trigonometric functions. This occurs when we must "complete the square" in order to rewrite the denominator in a rational integrand. We will review the method for completing the square in the context of evaluating a few integrals.

EX #4: Evaluate by completing the square.

A. $\int \frac{dx}{x^2 + 12x + 37}$

B. $\int \frac{dx}{\sqrt{4x - x^2}}$

C. $\int \frac{1}{x^2 + 6x + 25} \, dx$

D. $\int \frac{dx}{\sqrt{21 - 4x - x^2}}$

NOTES

Trigonometric Identities

There are times you might need to rewrite or rearrange integrals involving squared terms.

$\sin^2 x + \cos^2 x = 1$ $\quad 1 + \tan^2 x = \sec^2 x$ $\quad \cot^2 x + 1 = \csc^2 x$

$\sin 2x = 2 \sin x \cos x$ $\quad \cos 2x = \cos^2 x - \sin^2 x$

$\sin^2 x = \frac{1 - \cos 2x}{2}$ $\quad \cos^2 x = \frac{1 + \cos 2x}{2}$

Guidelines for Integrals Involving Only Products of Sine and Cosine:

$$\int \sin^m x \cos^n x \, dx$$

Note: This concept is not tested on the AP Exam

a) If m (the power of sine) is an odd integer, $(m = 2k + 1)$:

Save one sine factor and express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

Then substitute $u = \cos x$.

Note: If the powers of both sine and cosine are odd, either (a) or (b) can be used.

b) If n (the power of cosine) is an odd integer $(n = 2k + 1)$:

Save one cosine factor and express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

c) If m and n are even:

Use the half-angle identities:

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Sometimes helpful to use: $\sin x \cos x = \frac{1}{2} \sin 2x$

EX #5: Evaluate

A. $\int \sin^2 x \, dx$

B. $\int \sin^5 x \cos^2 x \, dx$

C. $\int \cos^3 x \, dx$

D. $\int \cos^3 x \sin^4 x \, dx$

Guidelines for integrals involving only products of Tangent and Secant:

$$\int \tan^m x \sec^n x \, dx$$

a) If m (the power of tangent) is an odd integer, $(m = 2k + 1)$:

Save a factor of $\sec x \tan x$ and express the remaining factors in terms of secant:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

Then substitute $u = \sec x$.

b) If n (the power of secant) is an even integer $(n = 2k)$:

Save a factor of $\sec^2 x$ and express the remaining factors in terms of tangent:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

EX #6: Evaluate

A. $\int \tan^3 x \, dx$

B. $\int \tan^3 x \sec^2 x \, dx$

Trigonometric Substitution

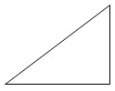
Let's look at another useful tool for transforming complicated integrals involving square roots into easier integrals.

Table of Trigonometric Substitutions

Expression in Integrand	Substitution		Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\cos^2\theta = 1 - \sin^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\sec^2\theta = 1 + \tan^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\tan^2\theta = \sec^2\theta - 1$

EX #7: Evaluate

A. $\int \frac{\sqrt{9 - x^2}}{x^2} \, dx$ Label the associated right triangle:



B. $\int \frac{1}{\quad} \, dx$ Label the associated right triangle:

NOTES

6.8 **Integration of Transcendental Functions Homework**

Name _____

Date _____ Period _____

Problems 1 -16, Find the indefinite integral for each of the following.

1. $\int (x^2 + 2^{-x})\,dx$

2. $\int \frac{1}{9 + (x-1)^2}\,dx$

3. $\int \frac{x^2 + 3x + 5}{x+1}\,dx$

4. $\int \frac{1}{\cos^2 x \tan x}\,dx$

5. $\int \sin^3 x \cos^4 x\,dx$

6. $\int \frac{2^{3x}}{1 + 2^{3x}}\,dx$

7. $\int \tan^3 x \sec^4 x\,dx$

8.

9. $\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$

10. $\int \sin^2 x \cos^3 x\,dx$

11. $\int x\left(3^{-x^2}\right)dx$

12. $\int e^{\sin \pi x} \cos \pi x\,dx$

13. $\int \frac{1}{\sqrt{3 - 4x^2}}\,dx$

14. $\int \frac{\cos^3 x}{\sqrt{\sin x}}dx$

15.

16. $\int \frac{x^2}{1 + x^2}dx$

Problems 17 – 24, Evaluate the definite integral.

17. $\int_1^2 \frac{1 - \cos t}{t - \sin t}\,dt$

18. $\int_0^{\pi/6} \frac{\cos\theta}{1 + 2\sin\theta}\,d\theta$

19. $\int_0^1 (5^x - 2^x)dx$

20. $\int_0^{\sqrt{\pi}} xe^{-\left(\frac{x^2}{2}\right)}\,dx$

21. $\int_0^1 \frac{4e^{2x}}{1 + e^{2x}}\,dx$

22.

23. $\int_0^{\ln 5} \frac{e^x}{1 + e^{2x}}\,dx$

24.

Problems 25 -28, Evaluate.

25. If $0 < k < \pi$, find k when $\int_0^k \sin 2x\,dx = 1$

A. $\frac{\pi}{4}$	B. $\frac{3\pi}{4}$	A. 2	B. 3
C. $\frac{\pi}{2}$	D. $\frac{3\pi}{2}$	C. 1	D. 5

27. $\int_e^{e^3} \frac{dx}{x \ln x}$

28. $\int_{\sqrt{2}}^2 \frac{x}{x^2 - 1}\,dx$

A. $\ln 3$	B. $\frac{1}{3}$	A. $1 - \sqrt{3}$	B. $\ln\sqrt{3}$
C. e	D. 3	C. $\ln\left(\frac{3}{2}\right)$	D. $\ln 3$

Problems 29 – 32, Evaluate using the proper trigonometric substitution.

29. $\int \frac{1}{x\sqrt{9 + x^2}}\,dx$

30. $\int \frac{1}{\sqrt{x^2 - 1}}\,dx$

31. $\int \frac{1}{x^2 + 36}\,dx$

Practice Sets

6.8-BC Integration of Transcendental Functions **Form A**

Name _____

Date _____Period _____

1. Find the indefinite integral:

$$\int e^{\tan(2\theta)} \sec^2(2\theta)\, d\theta$$

2. Find the indefinite integral:

$$\int \frac{1}{\sqrt{1-4x^2}}\, dx$$

3. Evaluate:

$$\int_e^{e^5} \frac{1}{x \ln x}\, dx$$

6.8-BC Integration of Transcendental Functions **Form B**

Name _____

Date _____Period _____

1. Find the indefinite integral:

$$\int e^{\sin(\pi\theta)} \cos(\pi\theta)\, d\theta$$

2. Find the indefinite integral:

$$\int \frac{3x^2}{\sqrt{1-x^6}}\, dx$$

3. E

$$\int$$

6.8-BC Integration of Transcendental Functions **Form C**

Name _____

Date _____Period _____

1. Find the indefinite integral:

$$\int e^{\tan(\pi\theta)} \sec^2(\pi\theta)\, d\theta$$

2. Find the indefinite integral:

$$\int \frac{3}{1+9x^2}\, dx$$

3. Evaluate:

$$\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\, dx$$

6.8-BC Integration of Transcendental Functions **Form D**

Name _____

Date _____Period _____

1. Find the indefinite integral:

$$\int e^{\sin(3\theta)} \cos(3\theta)\, d\theta$$

2. Find the indefinite integral:

$$\int \frac{1}{\sqrt{1-9x^2}}\, dx$$

3. Evaluate:

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

Daily Lesson Quiz