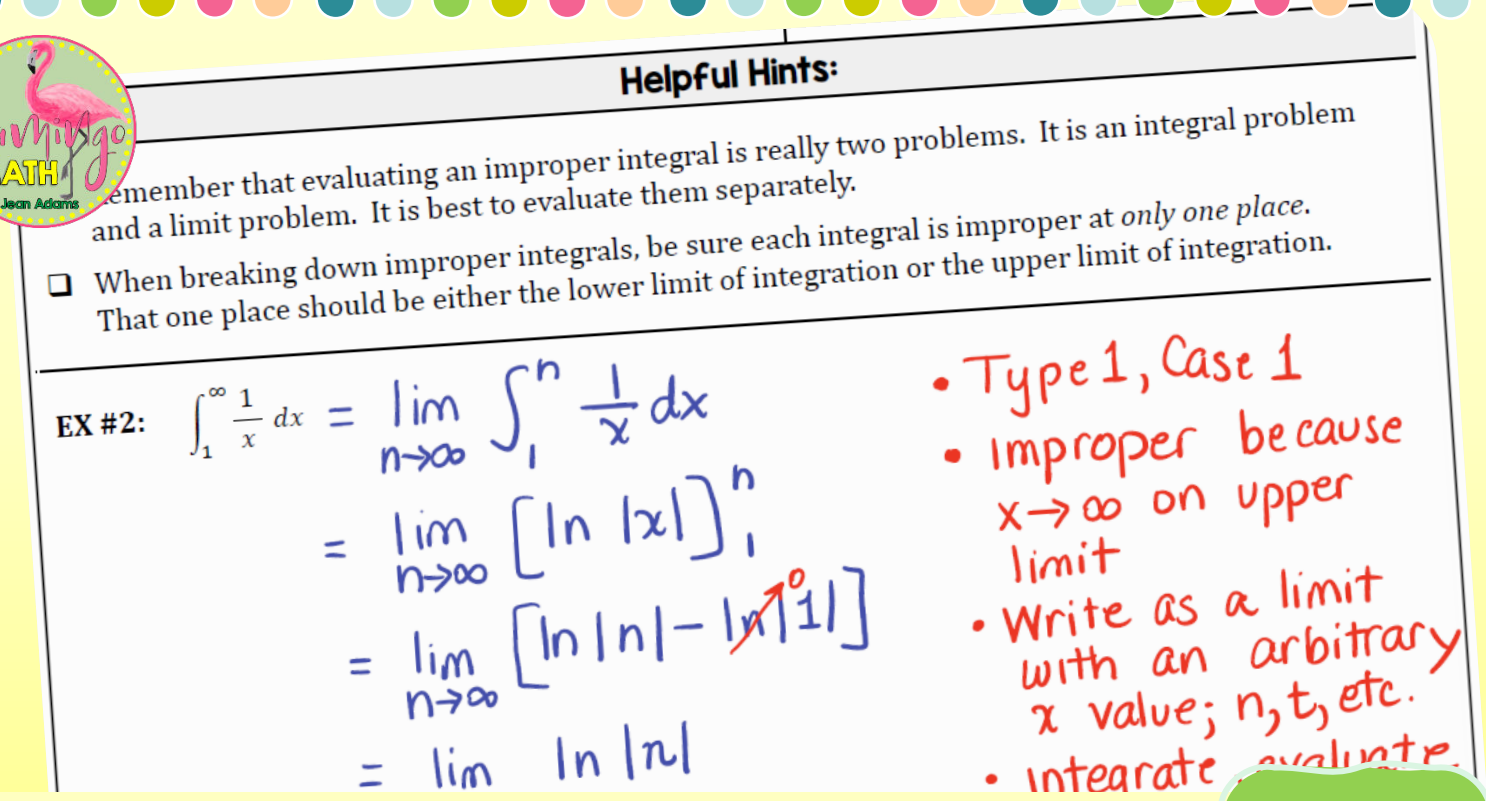
AP® Calculus-BC

Integration
Accumulation of Change



Helpful Hints:

Remember that evaluating an improper integral is really two problems. It is an integral problem and a limit problem. It is best to evaluate them separately.

- ❑ When breaking down improper integrals, be sure each integral is improper at *only one place.* That one place should be either the lower limit of integration or the upper limit of integration.

EX #2: $\int_1^\infty \frac{1}{x}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x}\,dx$

$= \lim_{n\to\infty} \left[\ln|x|\right]_1^n$

$= \lim_{n\to\infty} \left[\ln|n| - \ln|1|\right]$ (0)

$= \lim_{n\to\infty} \ln|n|$

- Type 1, Case 1
- Improper because $x \to \infty$ on upper limit
- Write as a limit with an arbitrary x value; n, t, etc.
- Integrate, evaluate

Guided Notes 6

Completely Editable

AP CALCULUS BC CURRICULUM

Scope and Sequence

UNIT 6	Integration – Accumulation of Change	17
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6–2	Sigma Notation and Riemann Sums	1
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6–4	Properties of Definite Integrals	1
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6–6	Fundamental Theorem of Calculus	1
6–7	Integration by Substitution	2
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AP CALCULUS BC CURRICULUM

Scope and Sequence

Unit 6: Integration - Accumulation of Change

Lesson	Skills & Learning Objectives
[6.1] Approximating Area with Riemann Sums	❑ Use Left, Right, and Midpoint Riemann Sums and Trapezoidal Rule to estimate area under a curve
[6.2] Sigma Notation and Riemann Sums	❑ Apply the properties of sequences to write integrals as a limit of right-hand sums.
[6.3] Accumulation Functions	❑ Use FTC to analyze functions, find displacement, total distance, and more.
[6.4] Properties of Definite Integrals	❑ Calculate a definite integral using areas and properties of definite integrals
[6.5] Antiderivatives and Indefinite Integrals	❑ Construct the general anti-derivative of a function through discovery and check approach.
[6.6] The Fundamental Theorem of Calculus	❑ Use the Fundamental Theorem of Calculus to find the definite integral of a function over an interval.
[6.7] Integration by Substitution	❑ Use pattern recognition and change of variables to find indefinite and definite integrals
[6.8] Integrals of Transcendental Functions	❑ Use rules of integration for transcendental functions, including inverse trigonometric rules.

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AP CALCULUS BC CURRICULUM

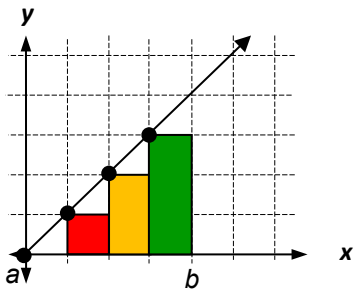
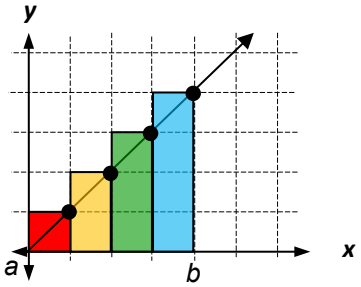
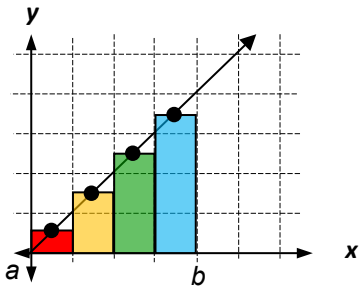
Scope and Sequence

Unit 6: Integration - Accumulation of Change

Lesson	Skills & Learning Objectives
[6.9] Integration by Parts	❑ Use the rules integration by parts, including the tabular method to find integrals.
[6.10] Partial Fractions	❑ Use partial fraction decomposition with linear factors and long division to find integrals.
[6.11] Improper Integrations	❑ Evaluate improper integrals and determine convergence or divergence.
[6.12 – 6.13]	❑ Select appropriate techniques for integration in mixed practice sets

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Three Types of Riemann Sums

Left-Endpoint (Lower sum) Inscribed Rectangles	Right-Endpoint (Upper Sum) Circumscribed Rectangles	Mid-Point (Average)
		

We can define the area A under the graph on the interval to be as follows.

DEFINITE INTEGRAL DEFINITION:

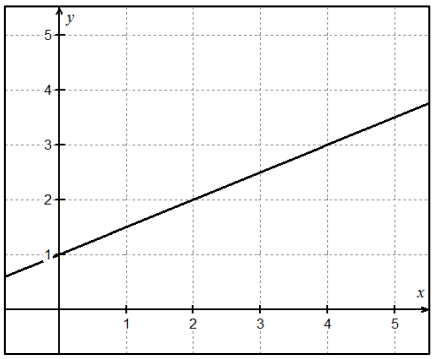
Let f be continuous on $[a, b]$ and $f(x) \geq 0$ for all x in the interval. Let $[a, b]$ be partitioned into n subintervals of equal length $\Delta x = \frac{b-a}{n}$. Then, the definite integral of f over $[a, b]$ is given by:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b f(x)dx$$

General Solution Method for Inscribed or Circumscribed Rectangles (Lower or Upper Sums):

1. Draw a rough graph of the function over the interval.
2. Use formula $\Delta x = \frac{b-a}{n}$ to determine each subinterval length.
3. Compute x-coordinates of rectangles at either left-end or right-end.
4. Compute the areas of each rectangle (inscribed or circumscribed).
5. Find the summation of the approximated areas of the rectangles.

EX #3: Find the area A under the graph of $f(x) = \frac{1}{2}x + 1$ on the interval $[0, 5]$ using geometry.



A Summary of the Procedure to Write Summation Notation

Find the width of each rectangle on the interval $[a, b]$	$\Delta x = \dfrac{b - a}{n}$
Find the right endpoint of each rectangle:	$x_k = a + \Delta x \cdot k$
Find the area of the k^{th} rectangle, where $f(x_k)$ is the height.	$\Delta x \cdot f(x_k)$
Right Riemann Sum, k-values go from 1 to n.	$\sum_{k=1}^{n} f(x_k) \cdot \Delta x$
Left Riemann Sum, k-values go from 0 to $n - 1$	$\sum_{k=0}^{n-1} f(x_k) \cdot \Delta x$

EX #4: Use the procedure above to write the expression that represents the approximation shown below.

A. $\sum_{k=1}^{6} \left(\dfrac{2}{1 + 0.5k} + 3 \right)$

B. $\sum_{k=0.5}^{4} \left(\dfrac{2}{1 + 0.5k} + 3 \right)$

C. $\sum_{k=1}^{6} \left(\dfrac{2}{1 + 0.5k} + 3 \right) (0.5)$

D. $\sum_{k=0}^{5} \left(\dfrac{2}{1 + 0.5k} + 3 \right) (0.5)$



A **definite integral** represents the area under the curve of a function, and a **Riemann sum** give us an approximation of that area. Imagine that we could create an infinite number of partitions (rectangles) in a Riemann sum, the approximation would get closer to the actual area as the subintervals in the Riemann sum increased. We can't take a Riemann sum with infinite rectangles, $n = \infty$ in our summation because infinity isn't a real number. However, if $R(n)$ represents the right Riemann sum approximation with n subdivisions, we can use **limits** when we want to take something to **infinity.**

Then, the exact value of the definite integral can be shown to be: $\lim_{n \to \infty} R(n)$

$R(n)$ a Right-Riemann Sum

$$R_n = \frac{b - a}{n} [y_1 + y_2 + y_3 + \cdots + y_n]$$

$R(n)$ written in Summation Notation:

$$Area = \int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{k=1}^{n} [f(x_k) \cdot \Delta x]$$

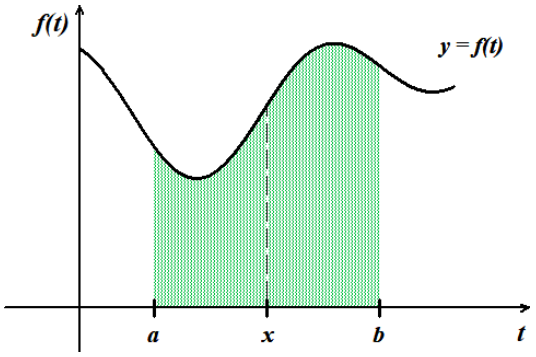
Lesson 6.3: Accumulation Functions

Topic 6.4: The Fundamental Theorem of Calculus and Accumulation Functions
Topic 6.5: Interpreting the Behavior of Accumulation Functions Involving Area

Given some function $y = f(x)$ is a continuous function on $[a, b]$, shown below. Suppose we define some function whose area under the curve between point a and some other point, x. We can denote the area under the curve as a definite integral. Let's define this as a new function $F(x)$, where x is in $[a, b]$:

$$F(x) = \int_a^x f(t)\,dt$$

So, for any value of x between a and b, this new function will give us the area under $f(t)$ between a and x. This is the Fundamental Theorem of Calculus Part I and helps to make the connection between integrals and derivatives. Namely, that every continuous function $f(x)$ has an antiderivative $F(x)$.



Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a, b]$ then, $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$

and it is differentiable on (a, b) also $g'(x) = f(x)$.

An alternate notation can be seen by taking the derivative of both sides of $F(x)$:

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

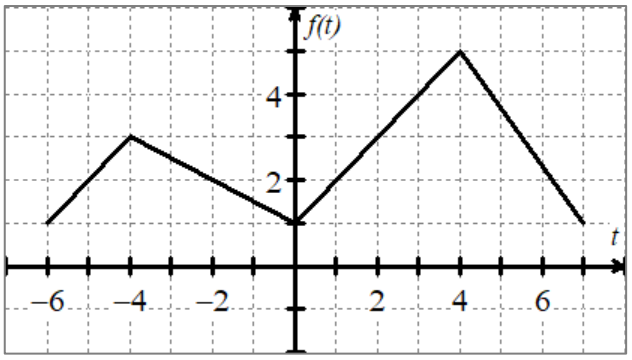
EX #1: The graph of f is shown below. Let $g(x) = \int_{-6}^x f(t)\,dt$. Evaluate the following:

A. $g(-3)$

B. $g(0)$

C. $g(4)$

D. $g(7)$



Lesson 6.5: Antiderivatives and Indefinite Integrals Basic Rules and Notation

Topic 6.8: Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

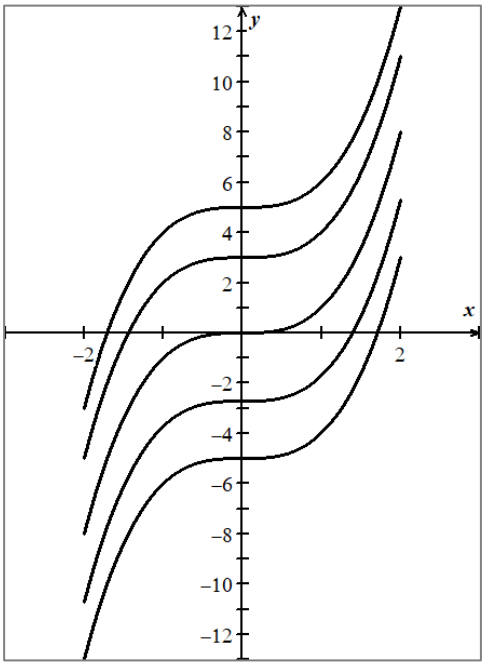
VOCABULARY AND DEFINITIONS:

1. A function $F(x)$ is an *antiderivative* of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is ***antidifferentiation***.
2. The family of all antiderivatives of a function $f(x)$ is the ***indefinite integral*** of f with respect to x and is denoted by $\int f(x)\,dx$.
3. If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$ is called the ***general solution*** and C is called the ***constant of integration*** (an arbitrary constant).

EX #1: We know that if a function f has an anti-derivative, $F(x)$, then there is a "*family of functions*" which share the same derivatives; and their antiderivatives will differ only by a constant, C.

A. State three functions F_1, F_2, F_3 whose derivatives are $f(x) = 3x^2$.

B. Explain the meaning of "+C" on your functions above.



EX #2: Applying Basic Rules

Given each derivative, below, find the general anti-derivative by Guess and Check.

A. $g'(x) = 2x$	**B.** $f'(t) = t^2$	**C.** $y' = \frac{1}{x^2}$
D. $\frac{df}{d\theta} = \cos\theta$	**E.** $F' = \frac{2}{3}x^{\frac{3}{4}}$	**F.** $f' = e^x$

Lesson 6.6: The Fundamental Theorem of Calculus and Definite Integrals

Topic 6.7: The Fundamental Theorem of Calculus and Definite Integrals

We have now seen the two major branches of calculus:

1) differential (tangent line problem)
2) integral (area problem)

Leibniz and Newton independently discovered a connection between these two branches, stated informally, that differentiation and (indefinite) integration are inverse operations. In an earlier lesson, we used the Fundamental Theorem of Calculus, Part I to define new functions, represent accumulation functions, and provide information about the antiderivative function.

THE FUNDAMENTAL THEOREM OF CALCULUS – PART I

If $f(x)$ is continuous on $[a, b]$ then, $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$

and it is differentiable on (a, b) also $g'(x) = f(x)$.

An alternate notation can be seen by taking the derivative of both sides of $F(x)$:

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

In words we can say: The Fundamental Theorem Part I says that if we define a function as the definite integral of another function, f, then the new function is an antiderivative of f.
Now we want to turn our attention to the FTC-Part II which will give us a tool for computing definite integrals in a quick and efficient manner.

Given that f represents the **RATE** at which the function $F(x)$ is changing, that is $f(x) = F'(x)$, we know that $\int_a^b f(x)dx$ represents the **TOTAL (NET) CHANGE IN F on the interval $[a, b]$**.

THE FUNDAMENTAL THEOREM OF CALCULUS – PART II

If a function f is continuous on a closed interval $[a, b]$ and $F(x)$ is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

If you like, you can also write this as: $\int_a^b g'(t)\,dt = g(b) - g(a)$

In words we can say: The **definite integral** of a function over $[a, b]$ is equal to the **NET change**, or **displacement**, in the antiderivative of f over the same interval. This is commonly known as the **Evaluation Theorem**.

Repeated Integration by Parts. **Do not switch choices for u and dv in successive applications.**

EX #8: $\int x^2 \cos x \; dx$

$u =$	$v =$
$du =$	$dv =$

$u_2 =$	$v_2 =$
$du =$	$dv =$

Tabular Method (Alternate Method for Integrating by Parts)

We can use this technique for repeated integration by parts. It works well for expressions of the form $x^n e^{kx}$, $x^n \sin(kx)$, and $x^n \cos(kx)$. Simply create three columns, D (find the derivative of your choice for u until the derivative equals 0) and I (for integrating your choice for dv). Then your solution is the sum of the signed products along the diagonal entries.

EX #9: Use the tabular method to integrate: $\int x^3 e^{2x} \; dx$

D	I	± 1
		+1
		−1
		+1
		−1
		+1
		−1

Lesson 6.II: Improper Integrals

Topic 6.13: Evaluating Improper Integrals

In our unit on integration, we discovered that a definite integral was defined as a limit of Riemann Sums. We used the Fundamental Theorem of Calculus to evaluate definite integrals. Now, we want to explore some integrals that do not meet the conditions of what we call a proper integral.

Definition of a Proper Integral

An integral expression of the form $\int_a^b f(x)\,dx$ is considered a proper integral if the limits and function are bounded. Proper integrals are always well-defined (the limit converges), that is:

- ❑ The function is defined and bounded on the interval $[a, b]$;
- ❑ The limits of integration are finite (not $\pm\infty$); and
- ❑ The function has only finitely many discontinuities in the interval.

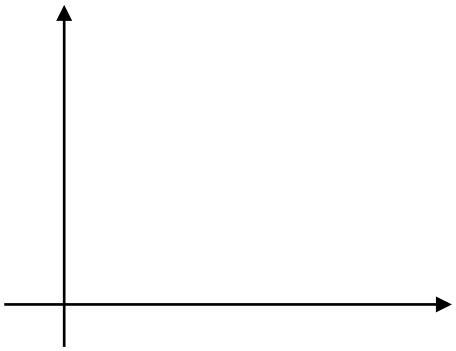
In this lesson, we investigate what happens when these conditions are not met. Integrals are considered **improper** if one of two things occur when defining the definite integral $\int_a^b f(x)\,dx$ on the interval $[a, b]$.

Type I:	Type 2:
Either a or b is ∞.	The region defined by $f(x)$ has an infinite discontinuity in $[a, b]$.

EX #1: Sketch the infinite region R, that lies under the curve $y = \frac{1}{x^2}$, above the x-axis. Then, shade the area to the right of the line $x = 1$ and ending to the right at some value $x = t$.

A. Solve the integral

$$A(t) = \int_1^t \frac{1}{x^2}\,dx$$



What does the answer tell you about the value of $A(t)$?

Find $\lim_{t\to\infty} A(t)$.