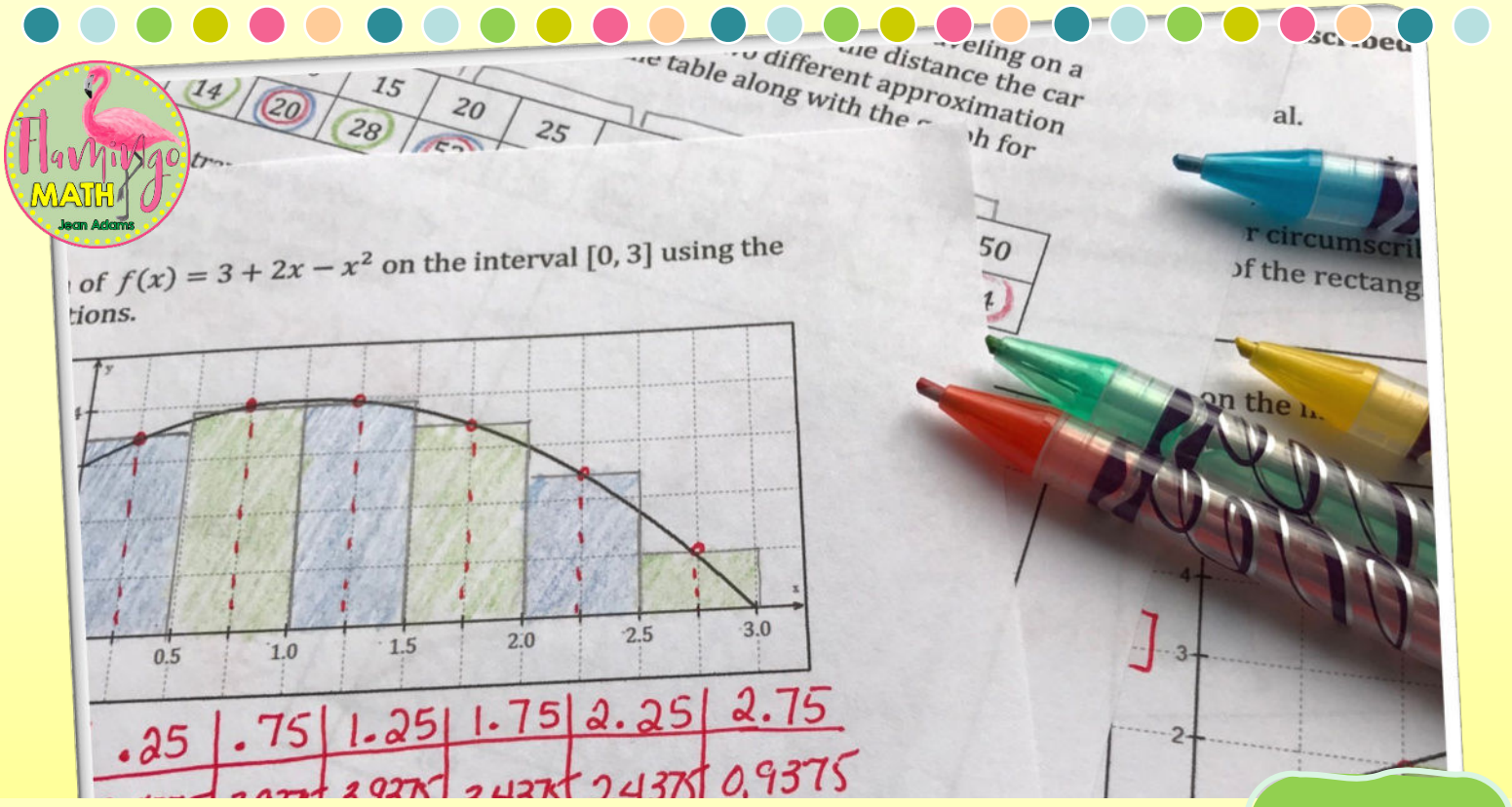
AP® Calculus

Integration

Accumulation of Change



Guided Notes 6

Completely Editable

AP CALCULUS AB CURRICULUM

Scope and Sequence

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AP CALCULUS AB CURRICULUM

Scope and Sequence

Unit 6: Integration - Accumulation of Change

Lesson	Skills & Learning Objectives
[6.1] Approximating Area with Riemann Sums	❑ Use Left, Right, and Midpoint Riemann Sums and Trapezoidal Rule to estimate area under a curve
[6.2] Sigma Notation and Riemann Sums	❑ Apply the properties of sequences to write integrals as a limit of right-hand sums.
[6.3] Accumulation Functions	❑ Use FTC to analyze functions, find displacement, total distance, and more.
[6.4] Properties of Definite Integrals	❑ Calculate a definite integral using areas and properties of definite integrals
[6.5] Antiderivatives and Indefinite Integrals	❑ Construct the general anti-derivative of a function through discovery and check approach.
[6.6] The Fundamental Theorem of Calculus	❑ Use the Fundamental Theorem of Calculus to find the definite integral of a function over an interval.
[6.7] Integration by Substitution	❑ Use pattern recognition and change of variables to find indefinite and definite integrals
[6.8] Integrals of Transcendental Functions	❑ Use rules of integration for transcendental functions, including inverse trigonometric rules.

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Finding Area Using Riemann Sums

Topic 6.2: Approximating Areas With Riemann Sums

What approach can we take if the rate of change function is not constant?

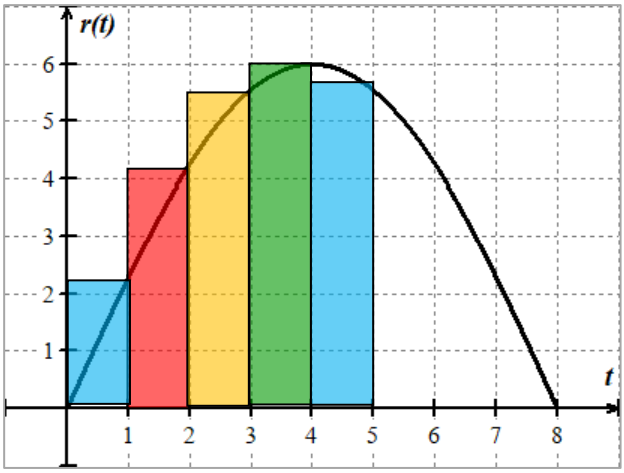
The two big ideas in calculus are the tangent line problem and the area problem. In the ***tangent line problem***, you saw how the limit process could be applied to find the slope of the tangent line (i.e. *the derivative*) at any point on the curve. A second classic problem in calculus is in finding the ***area under a curve*** that is bounded by the graph, the *x*-axis, and any two *x*-values.

A basic overview of "areas as limits." In the "limit of rectangles" approach, we take the area under a curve $y = f(x)$ above the interval $[a, b]$ by approximating a collection of **inscribed** or **circumscribed** rectangles in such a way that, the more rectangles used, the better the approximation. Finally, the number of rectangles is increased without limit and, bingo, we get the area! Now known as integration.

Specifically, we are interested in finding the area A of a region bounded by the x-axis, the graph of a ***nonnegative*** function $y = f(x)$ defined on some interval $[a, b]$.

Note: The requirement that f be nonnegative on $[a, b]$ means that no portion of its graph on the interval is below the x-axis. By using rectangles, we can see that there are three different ways of *approximating* the area for A. This method is commonly known as Riemann Sums.

EX #2: Consider a 30-gallon aquarium is being filled with water at a rate modeled by $r(t) = 6\sin\left(\frac{\pi}{8}t\right)$, where t is time, in minutes; and, $r(t)$ is gallons per minute. Find the volume of water in the tank after 5 minutes by using five rectangles that are 1-minute wide and $r(t)$ in height.



By using more rectangles with smaller widths, we could achieve a better approximation. In fact, if we considered taking this to a limit of accumulating an infinite number of rectangles we would arrive at the exact volume of water in the aquarium after 5 minutes. This is called the **definite integral** and is represented by the notation:

$$\int_0^5 r(t)dt$$

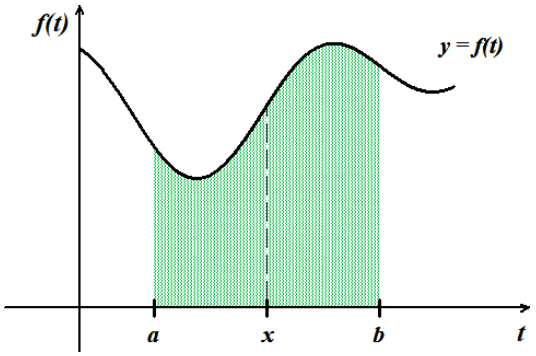
Lesson 6.3: Accumulation Functions

Topic 6.4: The Fundamental Theorem of Calculus and Accumulation Functions
Topic 6.5: Interpreting the Behavior of Accumulation Functions Involving Area

Given some function $y = f(x)$ is a continuous function on $[a, b]$, shown below. Suppose we define some function whose area under the curve between point a and some other point, x. We can denote the area under the curve as a definite integral. Let's define this as a new function $F(x)$, where x is in $[a, b]$:

$$F(x) = \int_a^x f(t)\,dt$$

So, for any value of x between a and b, this new function will give us the area under $f(t)$ between a and x. This is the Fundamental Theorem of Calculus Part I and helps to make the connection between integrals and derivatives. Namely, that every continuous function $f(x)$ has an antiderivative $F(x)$.



Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a, b]$ then, $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$

and it is differentiable on (a, b) also $g'(x) = f(x)$.

An alternate notation can be seen by taking the derivative of both sides of $F(x)$:

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

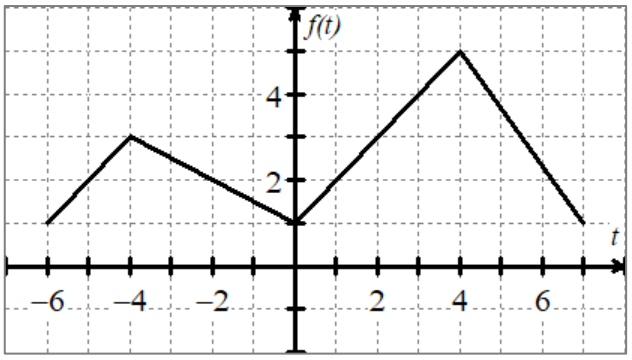
EX #1: The graph of f is shown below. Let $g(x) = \int_{-6}^x f(t)\,dt$. Evaluate the following:

A. $g(-3)$

B. $g(0)$

C. $g(4)$

D. $g(7)$

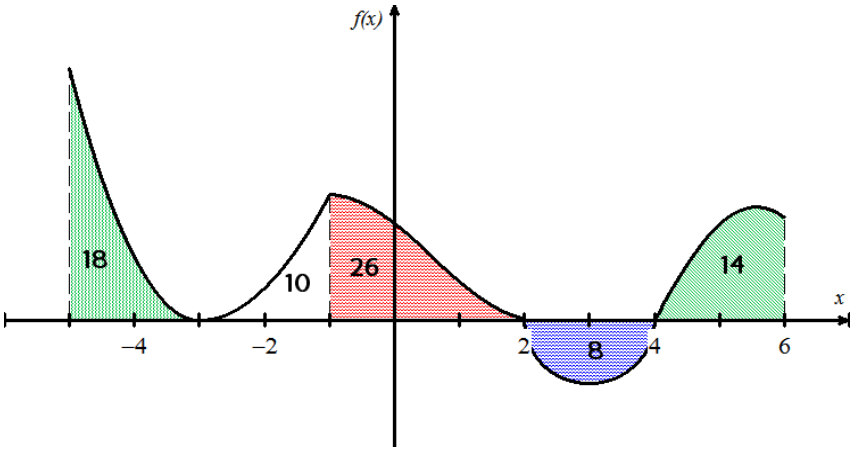


I. Describe the relationship between all $F(x)=\int_k^x f(t)\,dt$ as k takes on different values.

J. What point must lie on the graph of any given family member? Why?

K. As k takes on different values, the functions $F(x)=\int_k^x f(t)\,dt$ graph as _____ transformations of one another, each containing the particular point _____. Even though the lower bound changed is $F'(x) = f(t)$ **?** _____

EX #5: Shown below is the graph of the derivative f of some function F. Assume $F(-3) = -8$



A. $F(2) =$ _____

B. $F(6) =$ _____

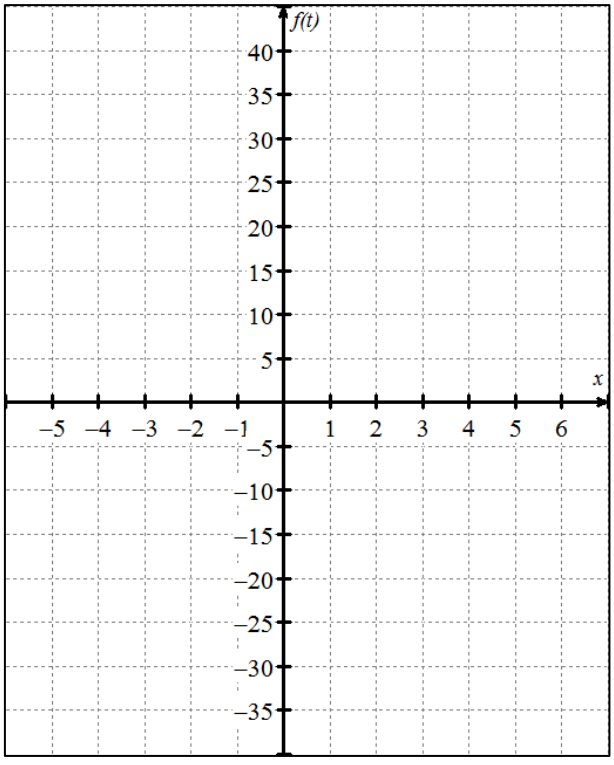
C. $F(-5) =$ _____

D. Since the area under f is positive on the interval $[-3,-1]$, we know that the graph of F must be _____ on the interval $[-3,-1]$.

E. Since the area under f accumulates at an increasing rate, we know that the graph of F must be _____ on the interval $[-3,-1]$.

F. The area under f on $[-1,2]$ accumulates at a _____ rate so F must be _____ on the interval $[-1,2]$.

G. Sketch a graph of F. Show extrema, inflection points, and horizontal tangents.



x	-5	-3	-1	2	4	6
$F(x)$						

Lesson 6.5: Antiderivatives and Indefinite Integrals Basic Rules and Notation

Topic 6.8: Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

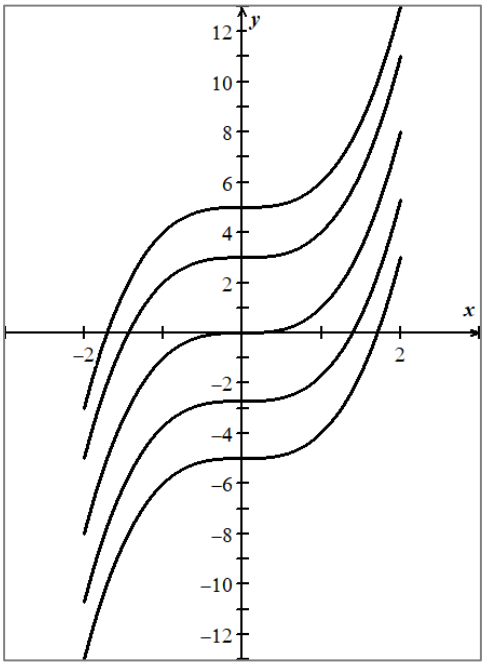
VOCABULARY AND DEFINITIONS:

1. A function $F(x)$ is an *antiderivative* of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is ***antidifferentiation***.
2. The family of all antiderivatives of a function $f(x)$ is the ***indefinite integral*** of f with respect to x and is denoted by $\int f(x)\,dx$.
3. If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$ is called the ***general solution*** and C is called the ***constant of integration*** (an arbitrary constant).

EX #1: We know that if a function f has an anti-derivative, $F(x)$, then there is a "*family of functions*" which share the same derivatives; and their antiderivatives will differ only by a constant, C.

A. State three functions F_1, F_2, F_3 whose derivatives are $f(x) = 3x^2$.

B. Explain the meaning of "+C" on your functions above.



EX #2: Applying Basic Rules

Given each derivative, below, find the general anti-derivative by Guess and Check.

A. $g'(x) = 2x$	**B.** $f'(t) = t^2$	**C.** $y' = \frac{1}{x^2}$
D. $\frac{df}{d\theta} = \cos\theta$	**E.** $F' = \frac{2}{3}x^{\frac{3}{4}}$	**F.** $f' = e^x$

Lesson 6.7: Integration By Substitution

Topic 6.9: Integrating Using Substitution

The key to *u*-substitution is finding a piece of the function whose derivative is also in the function. The derivative is allowed to be off by a coefficient, but otherwise must appear in the function itself.
The role of substitution in integration is comparable to the role of the Chain Rule in differentiation.

PROCEDURE FOR USING U-SUBSTITUTION:

1. Look for a piece of the function whose derivative is also in the function. If you're not sure what to use, try the denominator or something being raised to a power in the function.

2. Set *u* equal to that piece of the function and take the derivative with respect to the variable.

3. Use your *u* and *du* expressions to replace parts of the original integral, and your new integral will be much easier to solve.

The Substitution Method for Indefinite Integrals

Substitution is a method of integration designed to "undo" the chain rule. It generally involves 3 steps.

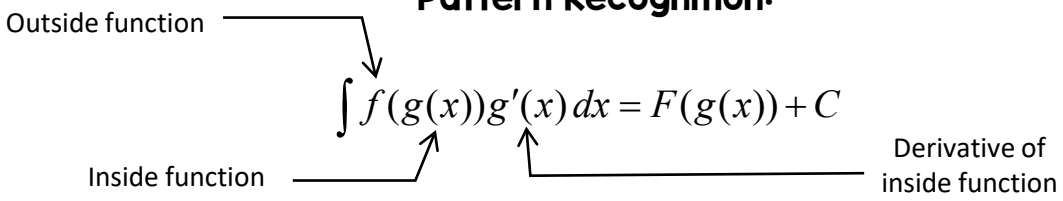
Given: $\int f\big(g(x)\big)\big(g'(x)\big)\,dx$

STEP 1: $= \int f(u)\,du$ Let $u = g(x)$, then $d(u) = g'(x)$

STEP 2: $= F(u) + C$ Find the antiderivative of f

STEP 3: $= F\big(g(x)\big) + C$ Substitute $g(x)$ back for u

Pattern Recognition:



THE GENERAL POWER RULE FOR INTEGRATION:

If g is a differentiable function of x, then

$$\int [g(x)]^n\, g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, n \neq -1$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1$$

Lesson 6.8: Integration of Transcendental Functions

Topic 6.9: Integrating Using Substitution

This is a collection of all transcendental functions and their corresponding integration rules.

Trigonometric Functions:

$$\int \sin u \, du = -\cos u + C \qquad \int \csc u \, du = -\ln|\csc u + \cot u| + C$$

$$\int \cos u \, du = \sin u + C \qquad \int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \tan u \, du = -\ln|\cos u| + C \qquad \int \cot u \, du = \ln|\sin u| + C$$

Logarithmic and Exponential Functions:

Let u be a differentiable function of x.

$$\int e^u \, du = e^u + C \qquad \int \frac{du}{u} = \ln|u| + C \qquad \int a^u \, du = \left(\frac{a^u}{\ln a}\right) + C$$

Inverse Trigonometric Functions:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C \qquad \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

Trigonometric Identities

There are times you might need to rewrite or rearrange integrals involving squared terms.

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad \cot^2 x + 1 = \csc^2 x$$

$\cos 2x = 2\cos^2 x - 1$ can also be expressed as $\cos^2 x = \frac{1 + \cos 2x}{2}$

$\cos 2x = 1 - 2\sin^2 x$ can also be expressed as $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$\sin 2x = 2\sin x \cos x$$