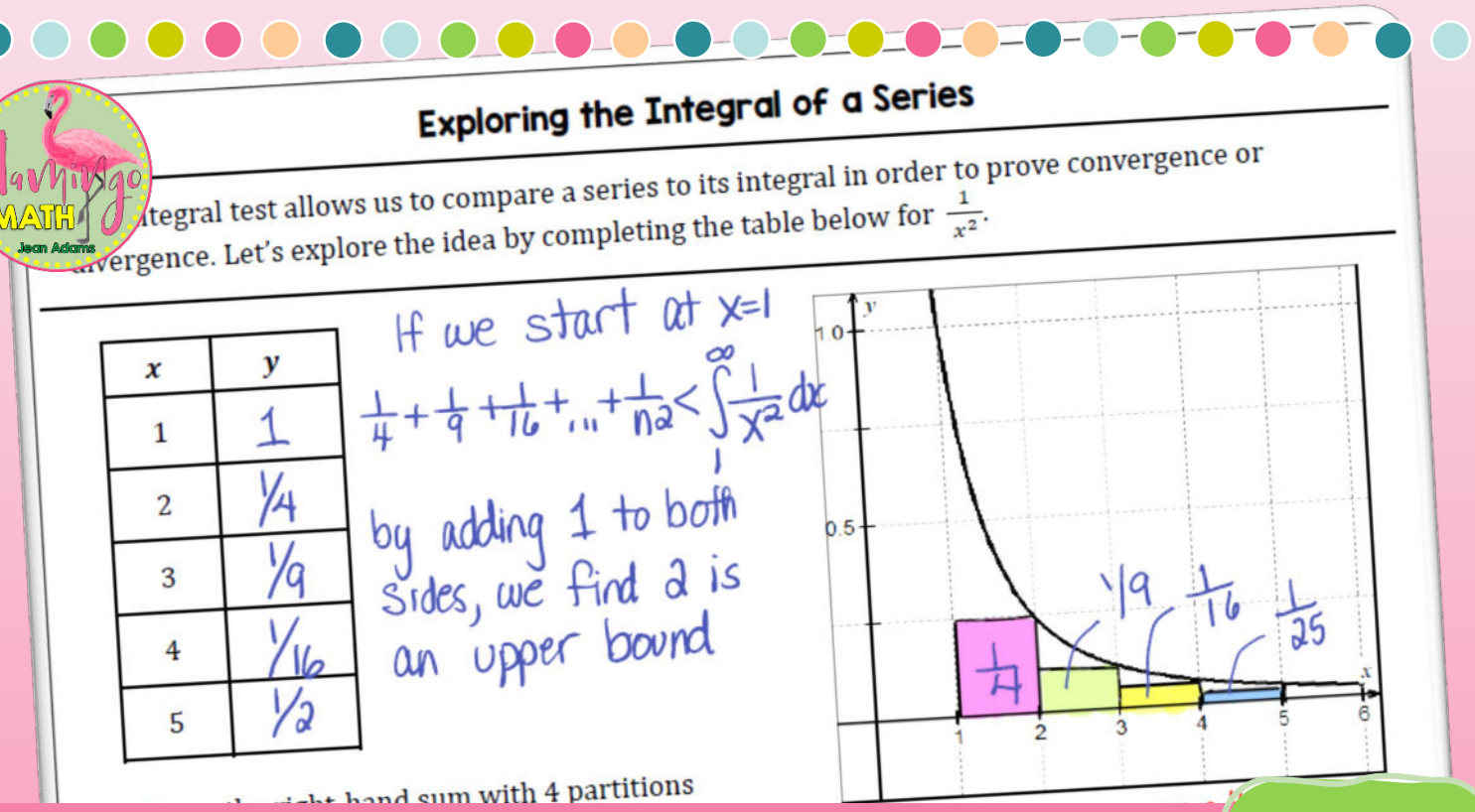
AP® Calculus-BC

Infinite Sequences and Series



Guided Notes 10

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AP CALCULUS BC CURRICULUM

Scope and Sequence

Unit 10: Series and Convergence

Lesson	Skills & Learning Objectives
[10.1] Series and Convergence	❑ Use properties of infinite geometric series and Nth-Term Test for Divergence
[10.2] Integral Test p-Series	❑ Use the Integral Test to determine convergence and divergence of infinite series ❑ Use properties of convergence for p-series
[10.3] Comparison of Series	❑ Use the Direct Comparison Test and Limit Comparison Test for convergence
[10.4] Alternating Series	❑ Use the Alternating Series Test for convergence
[10.5] Ratio and Root Tests	❑ Use the Ratio Test and Root Test to determine whether a series converges or diverges
[10.6] Taylor Polynomials	❑ Find Taylor and Maclaurin polynomial approximations of elementary functions
[10.7] Lagrange Error Bound	❑ Use Taylor polynomials to estimate values of functions
[10.8] Power Series & Intervals of Convergence	❑ Find the radius and interval of convergence of a power series, find endpoint convergence
[10.9] Taylor and MacLaurin Series	❑ Differentiate and integrate a Taylor or Maclaurin series
[10.10] Representing Functions by Power Series	❑ Find a geometric power series that represents a function

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Lesson 10.1: Series and Convergence

Topic 10.1-Defining Convergent and Divergent Infinite Series

An **infinite series** is defined as the summation of a sequence with an infinite number of terms. That is, if $\{a_n\}$ is an infinite sequence, then

Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Finding Partial Sums

We will find that when a sequence of partial sums converges, then the series will also converge. This lesson begins with a look at **geometric series** and **telescoping series**. The next lesson will explore **p-serie**s and **harmonic series**.

Infinite sums are called **series**. We want to investigate limits of sequences of numbers.

Series	Sequence	The value of a series is the limit of a sequence
$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$	$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots, \frac{2^n - 1}{2^n}, \cdots$	$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \frac{2^n - 1}{2^n}$

EX #1: Write the indicated partial sums S_n for the series

$$\sum_{k=1}^{n} \frac{1}{2^{k-1}}$$

A. S_1	**B.** S_2
C. S_3	**D.** S_n

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$ the ***n*th partial sum** is given by $S_n = a_1 + a_2 + \cdots + a_n$.

If the sequence of partial sums $\{S_n\}$ converges to S, then the series **converges.**

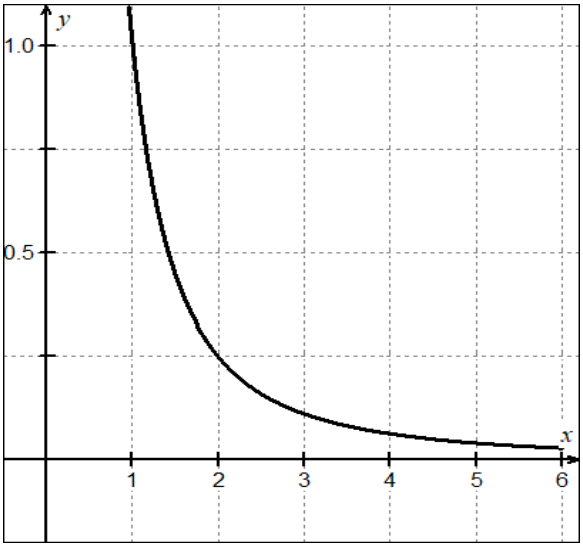
The limit S is called the **sum of the series.**

If $\{S_n\}$ diverges, then the series **diverges.**

Exploring the Integral of a Series

The integral test allows us to compare a series to its integral in order to prove convergence or divergence. Let's explore the idea by completing the table below for $\frac{1}{x^2}$.

x	y
1	
2	
3	
4	
5	



1. Draw the right-hand sum with 4 partitions over $[1, 5]$. Find the value.

   RHS_4 = _____

2. Find the 5th partial sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$

3. $\int_1^5 \frac{1}{x^2} dx =$

The Integral Test

To use this test, f MUST BE **CONTINUOUS, POSITIVE,** and **DECREASING** on the interval $[1, \infty)$.

Also, $a_n = f(n)$.

☐ If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

☐ If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note, the most common use is the *p*-Series.

Lesson 10.4: Alternating Series

Topic 10.5: Harmonic Series and p-Series

This section will introduce us to alternating series that contain both positive and negative terms. Let's start by revisiting the harmonic series.

Harmonic Series Exploration

Given the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

1. Consider the idea that you wanted to have the signs alternate on each term. What would you need to do to the formula to cause this to happen?

2. Use the number line below to plot the sum of your new series.



Topic 10.7: Alternating Series Test for Convergence

So, by definition, an alternating series is a series whose terms are alternately positive and negative. Alternating series can occur in two ways: either the odd terms are negative, or the even terms are negative.

Alternating Series Test

The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge if the following **conditions are met**:

1. Every a_n is positive **2.** $\lim_{n\to\infty} a_n = 0$ **3.** $a_{n+1} < a_n$, for all n

Verbally Speaking:

A series converges if its terms:

1. Alternate in sign,
2. Decrease in magnitude; and
3. Have a limit of zero.

Summary of Tests for Sequences and Series

Test	Series	Converges	Diverges	Comments
Nth Term	$\sum_{n=0}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	NOT for convergence
Geometric Series (GST)	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert > 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (a_n - a_{n+1})$	$\lim_{n\to\infty} a_n = L$		Sum: $a_1 - L$
p-Series Test	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	If $p = 1$ this is a divergent harmonic series
Alternating Series Test (AST)	$\sum_{n=1}^{\infty} (-1)^n a_n$ $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$0 < a_{n+1} \leq a_n$ $\lim_{n\to\infty} a_n = 0$		Remainder Error Bound: $\lvert S - S_n\rvert = \lvert R_N\rvert$ $\lvert R_N\rvert \leq a_{n+1}$
Integral Test f is positive, continuous, and decreasing $a_n = f(n)$	$\sum_{n=1}^{\infty} a_n$	$\int_1^{\infty} f(x)\,dx$ converges, then series converges	$\int_1^{\infty} f(x)\,dx$ diverges, then series diverges	Remainder $0 \leq R_n \leq \int_n^{\infty} f(x)dx$
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$ or $= \infty$	Inconclusive if $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$ or $= \infty$	Inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$
Direct Comparison Test (DCT)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ Diverges	
Limit Comparison Test (LCT)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	L must be finite and positive. Then either both series converge or both series diverge.

Lesson 10.6 : Intro to Taylor Polynomials

Topic 10.11: Finding Taylor Polynomial Approximations of Functions

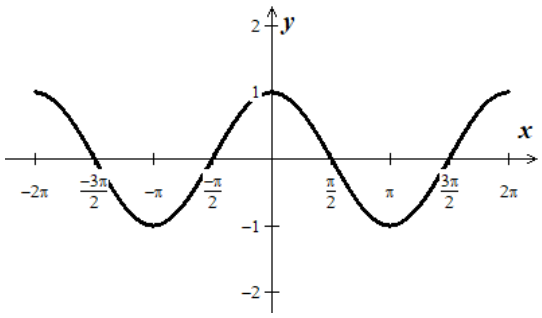
In this exploration activity you will learn how to construct a Taylor Polynomial from scratch.

A Taylor Polynomial Exploration

1. Find the equation of the tangent line of $y = \cos x$ at $x = 0$.

   _____

2. Sketch your linear approximation on the graph at right.



3. Notice that our approximation has the same y-value and slope as the original function. What other property could be used so that it matches at the point $x = 0$?

   _____

4. What degree polynomial would this approximation need to be? _____

A Quadratic Approximation at $x = 0$

	Quadratic Approximation	Function we are approximating
$f(x)$		
$f'(x)$		
$f''(x)$		

5. What is the center of your approximation? $x =$ _____

	Quadratic Approximation	Function we are approximating
$f(0)$		
$f'(0)$		
$f''(0)$		

6. Use the table above to find values for $a =$ _____ $b =$ _____ $c =$ _____

7. The quadratic approximation for $\cos x$ is: _____

8. Sketch your approximation on the graph above.

In the exploration, our polynomial approximation matched the value of $\cos x$, it's first and second derivative at the center, $x = 0$. Let's try to make that match more precise by using an even higher order approximation centered at $x = 0$.

Lesson 10.8: Power Series and Interval of Convergence

Topic 10.13: Radius and Interval of Convergence of Power Series

We have studied infinite series and learned about Taylor polynomial approximations. In this lesson and the next, we will explore how many functions, such as e^x, $\cos(x)$, $\sin(x)$ can be rewritten as polynomials with an infinite number of terms. **Taylor and Maclaurin series** are actually specific kinds of power series. A power series has a variable in it.

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is called a **power series.** While an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \cdots + a_n (x-c)^n + \cdots$$

is called a **power series centered at *c*,** where c is a constant.

Taylor Polynomials centered about x = c

$$f(x) \approx P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^n(c)}{n!}(x-c)^n$$

EX #1: Some important details to consider. Some functions shown below will not generate a Taylor polynomial at $x = c$ because the function or one of its derivatives don't exist there. Determine which functions present problems and explain why.

A. $y = \sqrt{x-1}$ centered at $x = 2$	**B.** $y = \ln(x-1)$ centered at $x = 1$
C. $y = cot\ x$ centered at $x = 0$	**D.** $y = \sqrt{x+1}$ centered at $x = -1$
E. $y = \sqrt{x-1}$ centered at $x = 0$	**F.** $y = \frac{1}{x-3}$ centered at $x = 4$