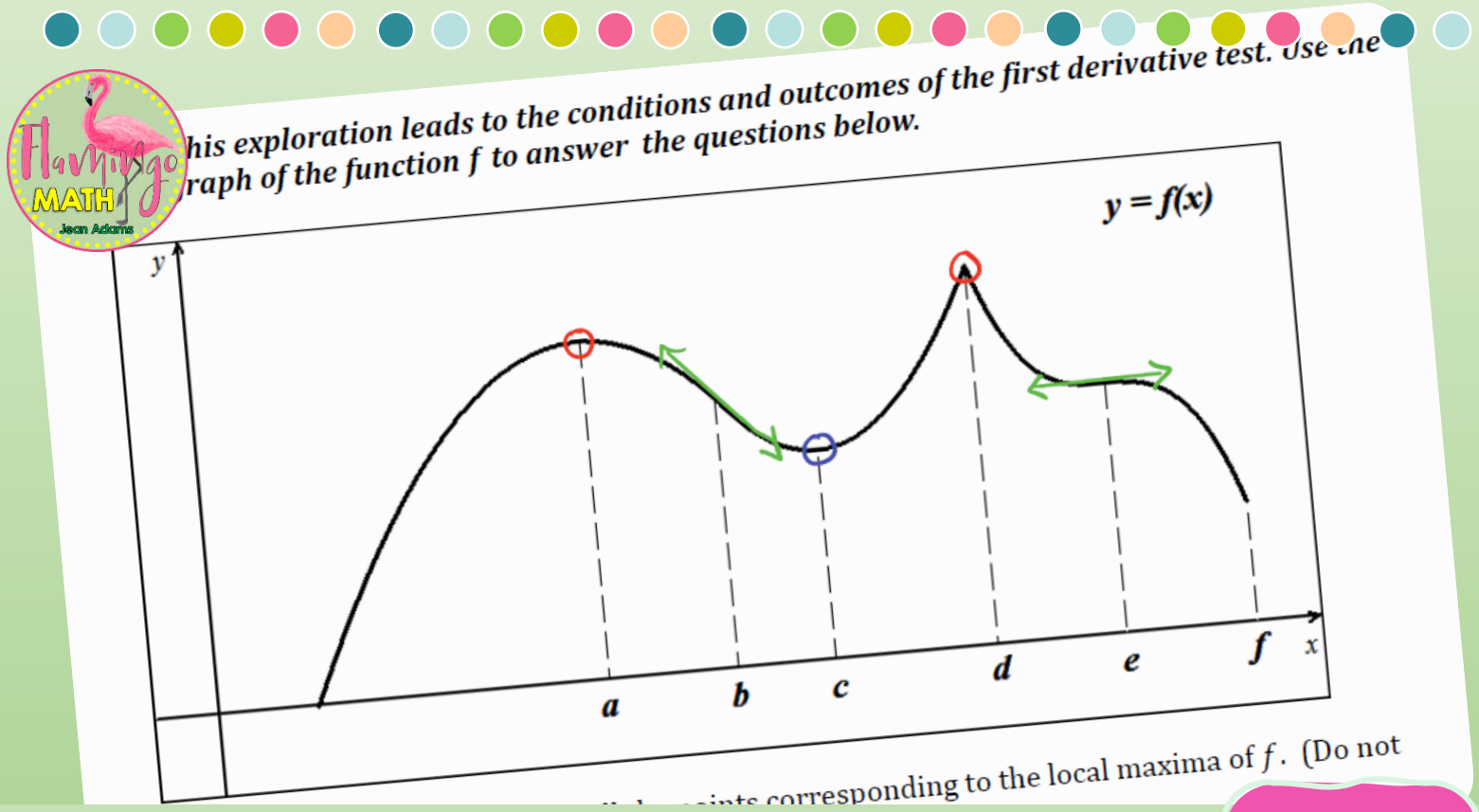
AP® Calculus

First Derivative Test

Increasing and Decreasing Functions



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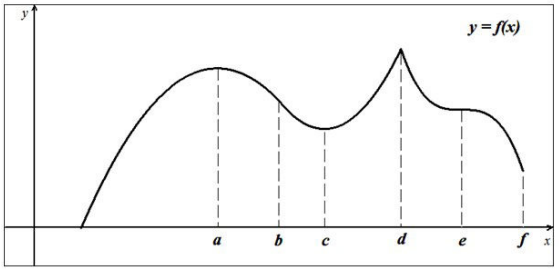
5.3

Lesson 3: The First Derivative Test
Increasing and Decreasing Functions

Topic 5.3: Determining Intervals on Which a Function Is Increasing or Decreasing

Derivatives can be used to classify relative extrema as either relative minima or relative maxima. As we begin to focus on analyzing functions, we can determine some predictable and specific behaviors for functions in our calculus journey through "the mathematics of change."

EX #1: ***This exploration leads to the conditions and outcomes of the first derivative test. Use the graph of the function f to answer the questions below.***



1. Using a ***RED*** colored pencil, circle all the points corresponding to the local maxima of f. (Do not include endpoints.)

   A. State the x-values where the local maxima of f occur. _____

   B. At each maximum value, f changes from _____ to _____

   C. At each maximum value, f' changes from _____ to _____

2. Using a ***BLUE*** colored pencil, circle all the points corresponding to the local minima of f. (Do not include endpoints.)

   A. State the x-values where the local minima of f occur. _____

   B. At each minimum value, f changes from _____ to _____

   C. At each minimum value, f' changes from _____ to _____

3. At each relative (local) extreme value, $f'(x)$, the derivative of f, is either _____ or _____

4. Using a ***GREEN*** colored pencil, sketch a tangent line, if possible, at the following points and determine their sign.

   A. At point b, $f'(b)$ is _____. Describe the change in the behavior of the graph of f at this value. _____

   B. At point d, $f'(d)$ is _____. Describe the change in the behavior of the graph of f at this value. _____

   C. At point e, $f'(e)$ is _____. Describe the change in the behavior of the graph of f at this value. _____

   D. Of these three points which two are points of inflection? _____

5. If a function f has a relative extremum at a point, then the derivative at the point is either zero or undefined. Is the converse of the statement also true? _____ What point(s) justifies your answer? _____

_____

Definitions: Increasing and Decreasing Functions

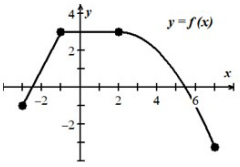
- ❑ A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- ❑ A function f is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

EX #2: The graph of $y = f(x)$, shown at right, is defined on $[-3, 7]$. List the *open intervals* over which the function is increasing, decreasing, or constant.

A. $f(x)$ is increasing on _____

B. $f(x)$ is decreasing on _____

C. $f(x)$ is constant on _____



NOTE: A function is said to be MONOTONIC on an interval if its first derivative (which need not be continuous) does not change sign.

How can we find the values of x on the graph of a function where it is increasing, decreasing, or constant? Some important ideas come to mind here. If $f(x)$ is a **continuous function**, then the derivative can only change signs at a **critical value**. And, if $f(x)$ is a **discontinuous function**, then the derivative could change signs at a **critical value** or at a **discontinuity**.

How to Find Intervals of Increase and Decrease

Let f be a continuous function on the closed interval $[a, b]$. To find the intervals on which f is increasing or decreasing:

1. **Find the critical values** of f, including discontinuities, in the interval (a, b).
2. Create a **number line chart** using these critical numbers.
3. Choose ***x*-values in between the critical numbers** to test the sign of f' on your sign chart.

NOTE: Your sign chart is NOT A VALID JUSTIFICATION. You MUST write a clear and concise statement communicating your mathematical language related to sign changes of the derivative. ***DO NOT USE PRONOUNS!***

For example,
"The function g is increasing on the interval $(-2, 5)$ ***since*** $g'(x) > 0$."

Topic 5.3: Determining Intervals on Which a Function Is Increasing or Decreasing

EX #3: Find the open intervals on which $f(x) = \frac{2}{3}x^3 - x^2 - 4x + 2$ is increasing and or decreasing. Justify.

When presented with a continuous function, knowing ***that*** the sign of the derivative changes at a point $(c, f(c))$, as well as, knowing ***how*** the sign changes will give us insight into the existence of extrema.

NOTES

Topic 5.4: Using the First Derivative Test to Determine Relative (Local) Extrema

FIRST DERIVATIVE TEST:

Let c be a critical number of a function f that is continuous on an open interval. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a ***relative minimum*** of f.
2. If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a ***relative maximum*** of f.
3. If $f'(x)$ does not change signs at c, then $f(c)$ is neither a ***relative minimum*** nor a ***relative maximum*** of f.

We will explore how to find points of inflection and intervals where a function is concave up or concave down in the next lesson. The following chart will be helpful in summarizing and ***using correct vocabulary*** when justifying conditions:

Function Behavior				
$f(x)$	∪	∩	↗	↘
$f'(x)$	↗	↘	+	−
$f''(x)$	+	−		

Symbol	Vocabulary
+	Positive
−	Negative
↗	Increasing
↘	Decreasing
∪	Concave Up
∩	Concave Down

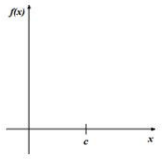
EX #4: Given the function: $f(x) = x^4 - 8x^2 + 1$

A. Find the open intervals on which $f(x)$ is increasing and/or decreasing. Justify.

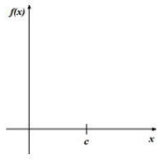
B. Determine the x-values of any local extrema of $f(x)$. Justify.

EX #5: For each statement below, sketch a function that represents the results of the first derivative test for a continuous function $y = f(x)$ with a critical number at $x = c$.

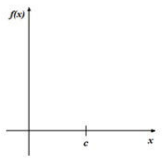
A. $f(x)$ has an absolute minimum at $x = c$, because $f'(x)$ changes sign from negative to positive at $x = c$.



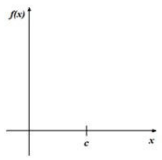
B. $f(x)$ has a local maximum $x = c$, because $f'(x)$ changes sign from positive to negative at $x = c$.



C. $f(x)$ has no local extrema at $x = c$ because $f'(x) > 0$ for all $x > 0$ and is strictly monotonic.



D. Since $f'(x)$ is negative immediately on either side of $x = c$, the function $f(x)$ has neither a local maximum nor local minimum at $x = c$.



EX #6: Find the exact values of any relative extrema of the function $g(x) = \frac{1}{2}x - \sin x$ on the interval $[0, 2\pi]$. Justify.

ANSWER THE QUESTION BEING ASKED . . . Are you looking for extrema (Y-VALUES) or the location where the extrema occurs (X-VALUES)?

EX #7: Find the relative extrema for $f(x) = (x^2 - 4)^{\frac{2}{3}}$. Justify.

EX #8: Find the x-coordinates of the relative extrema for $h(x) = \sqrt[3]{x^2}(x - 2)$

EX #9: Given $g(x) = \frac{x^2 - 3}{x - 2}$, **(i)** Find the intervals of increasing/decreasing and **(ii)** Find the x-coordinates of the relative extrema.

NOTES

5.3 **Increasing, Decreasing and the First Derivative Test Homework**

Name _____

Date _____ Period _____

Problems 1-4; Identify the open intervals on which the function is increasing or decreasing.

1. $f(x) = 8x - x^3$

2. $g(x) = x\sqrt{9 - x^2}$

3. $h(x) = x + 2\sin x \qquad [0, 2\pi]$

4. $f(x) = e^{-x} + e^{2x}$

Problems 5 – 9; Find the open intervals on which the function is increasing or decreasing and locate all relative extrema. Check your results with a graphing utility.

5. $f(x) = x^3 - 6x^2 + 15$

6. $f(x) = \frac{x^3}{3} - \ln x$

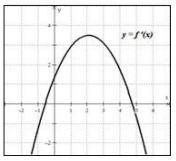
7. $f(x) = \sqrt[3]{x} - 1$

8. $f(x) = \frac{x^2}{x+1}$

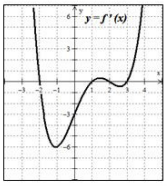
9. $f(x) = (x-2)e^x$

Problems 10- 11; Use the graph of f' to identify the interval(s) on which f is increasing or decreasing and estimate the value(s) of x at which f has a relative maximum or minimum.

10.

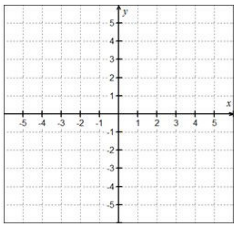


11.

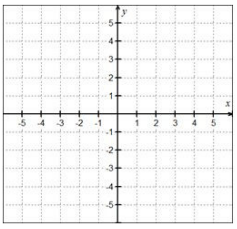


Problems 12 – 13; Sketch the graph of a continuous function f that satisfies all the stated conditions.

12. $f(0) = 1;\ f(2) = 3;\ f'(0) = f'(2) = 0;$
$f'(x) < 0$ on $(-\infty, 0) \cup (2, \infty)$
$f'(x) > 0$ on $(0, 2)$



13. $f(1) = f(5) = 0;\ f'(3)$ does not exist
$f'(x) < 0$ if $x < 3;\ f'(x) > 0$ if $x > 3$
$f(x)$ is concave down if $x \neq 3$



B) $g(x) = f(x - 8)$ $\quad g'(1)$ ☐ 0

C) $g(x) = f(2x) - 4$ $\quad g'(-4)$ ☐ 0

Problems 15 – 18: use the First Derivative Test to complete the following.

15. For what values of x is the function decreasing? $f(x) = 16 + 24x + 3x^2 - x^3$

A. $-2 < x < 4$

B. $-4 < x < 2$

C. $x < -2$ or $x > 4$

D. $x < -4$ or $x > 2$

16. Let $f(x) = x\sqrt{4 - x^2}$. Find all relative extrema of f.

17. Let $g(x) = 2xe^{-x}$; for $x \geq 0$.

A. Find where g is increasing and decreasing, Justify.

B. Determine the extrema of $g(x)$ on $[0, \infty)$. Justify.

Practice Sets

5.3 First Derivative Test **Form A**

Name _____

Date _____Period _____

$f(x) = x + 2\cos(x)$ on $[0, \pi]$

1. Find the critical numbers of $f(x)$

2. Find the open intervals on which the function is increasing or decreasing.

3. Locate all absolute extrema.

Grade:

5.3 First Derivative Test **Form B**

Name _____

Date _____Period _____

$f(x) = -x^3 - 6x^2 - 9x - 2$

1. Find the critical numbers of $f(x)$

2. Find the open intervals on which the function is increasing or decreasing.

3. Locate

Grade:

5.3 First Derivative Test **Form C**

Name _____

Date _____Period _____

$f(x) = x + 2\sin(x)$ on $[0, 2\pi]$

1. Find the critical numbers of $f(x)$

2. Find the open intervals on which the function is increasing or decreasing.

3. Locate all absolute extrema.

Grade:

5.3 First Derivative Test **Form D**

Name _____

Date _____Period _____

$f(x) = x^3 - 3x^2 - 24x + 8$

1. Find the critical numbers of $f(x)$

2. Find the open intervals on which the function is increasing or decreasing.

3. Locate all absolute extrema.

Grade:

Daily Lesson Quiz