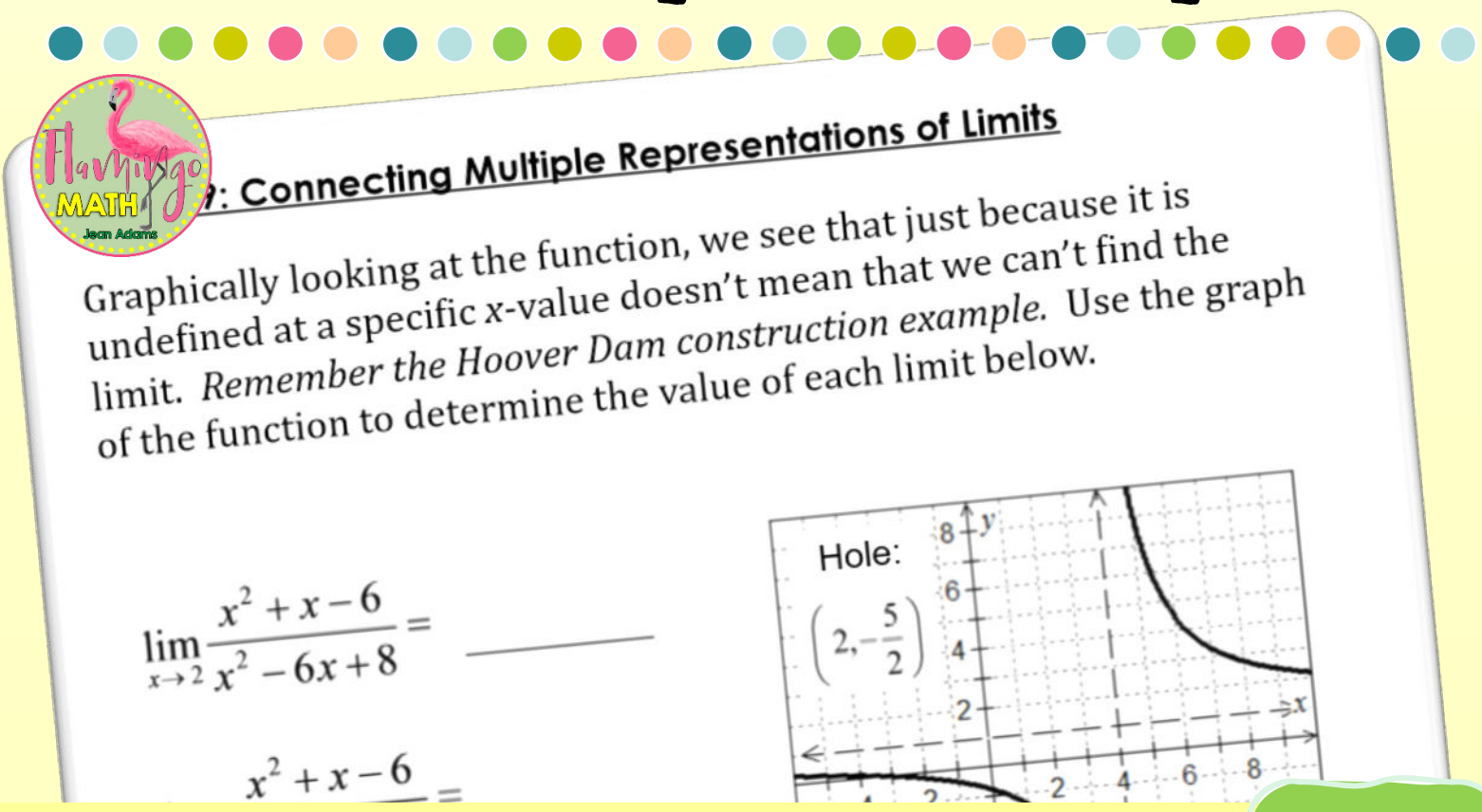
AP® Calculus

Finding Limits Analytically



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1.5

Lesson 5: Finding Limits by Analytic Methods

Topic 1.6: Determining Limits Using Algebraic Manipulation

If you only observe the graph of a function, it can be misleading when finding the limit of a function. In this lesson we will explore how to find limits using algebraic techniques and limit theorems.

You will learn to analyze limits by the following methods:

Methods to Analyze Limits:

1. Direct substitution.
2. Basic Limit Theorems
3. Factor, cancellation technique. Then go back to step 1.
4. The conjugate method, rationalize the numerator. Then, go back to step 2.
5. Special trig limits of $\lim_{x\to 0}\frac{\sin x}{x}=1$ or $\lim_{x\to 0}\frac{1-\cos x}{x}=0$
6. L'Hospital's Rule (presented in Unit 3)

Substitution Theorem

If f is a polynomial function or rational function, then $\lim_{x\to c} f(x)=f(c)$ provided that if f is a rational function the value of the denominator does not equal 0.

NOTE: ***Always try DIRECT SUBSTITUTION first.*** *If you get* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ *the goal will be to simplify the expression using algebraic techniques and then try substitution again.*

EX #1: Find each of the following limits analytically using direct substitution.

A. $\lim_{x\to 2}(3x^2-5x+4)$

B. $\lim_{x\to 2}\frac{x^2+1}{x+1}$

C. $\lim_{x\to e}\frac{\ln x}{3x}$

D. $\lim_{x\to 4}\sqrt[3]{x+4}$

E. $\lim_{\theta\to\frac{\pi}{6}}\sin 2\theta$

F. $\lim_{x\to 5}\log_3(x+4)$

REMEMBER: When you get indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ your goal is to algebraically manipulate the expression in an effort to remove the point(s) of discontinuity. Then, try direct substitution again. Let's explore a few techniques.

Finding One-Sided Limits

Recall, in our example there is a vertical asymptote at $x=4$.
How can we determine the behavior of a function at this value of x?

$\lim_{x\to 4^+}\frac{x^2+x-6}{x^2-6x+8}=$

As $x\to 4^+$, pick a value to the right of 4, then analyze the simplified function:

$\lim_{x\to 4^+} f(x)=$

$\lim_{x\to 4^-}\frac{x^2+x-6}{x^2-6x+8}=$

As $x\to 4^-$, pick a value to the left of 4, then analyze the simplified function:

$\lim_{x\to 4^-} f(x)=$

EX #2: Finding limits analytically of piecewise functions. Show your algebraic steps.

A. $\lim_{x\to 1} g(x)$ given, $g(x)=\begin{cases}x^3+1, & x>1\\ x+1, & x\le 1\end{cases}$

B. $\lim_{x\to 2} h(x)$ given, $h(x)=\begin{cases}x^2-4x+7, & x\le 2\\ -x^2+4x-1, & x>2\end{cases}$

Topic 1.10: Exploring Types of Discontinuities

What is the process for finding discontinuities of a rational function from Pre-calculus?

You can perform the same algebraic analysis to find the limit of the removable, or point discontinuities and the non-removable, or infinite discontinuities using what we will call the **Factoring Method or Cancellation Technique**.

$\lim_{x\to 2}\frac{x^2+x-6}{x^2-6x+8}$

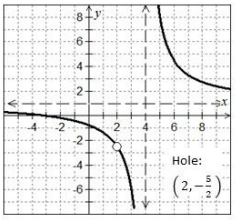
Topic 1.9: Connecting Multiple Representations of Limits

Graphically looking at the function, we see that just because it is undefined at a specific x-value doesn't mean that we can't find the limit. *Remember the Hoover Dam construction example.* Use the graph of the function to determine the value of each limit below.

$\lim_{x\to 2}\frac{x^2+x-6}{x^2-6x+8}=$ _____

$\lim_{x\to 4^+}\frac{x^2+x-6}{x^2-6x+8}=$ _____

$\lim_{x\to 4^-}\frac{x^2+x-6}{x^2-6x+8}=$ _____



Algebraically Finding Limits of Functions at Undefined Values

Consider what happens when you try to evaluate this limit using direct substitution.

Topic 1.7: Selecting Procedures for Determining Limits

EX #3: The Factoring or Cancellation Technique

A. $\lim_{x\to -4}\frac{2x^2+7x-4}{x^2-x-20}$

B. $\lim_{x\to 3}\frac{x^3-27}{x^2+x-12}$

EX #4: The LCM/LCM Method for Complex Fractions

A. $\lim_{x\to 0}\frac{x}{\frac{1}{4}+\frac{1}{x-4}}$

B. $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}$

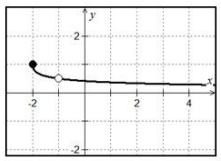
NOTES

EX #5: The Rationalization Technique or Conjugate Method

The technique of rationalization can be used to find the limit when there is a radical in the numerator or denominator.

A. The graph of $g(x) = \frac{\sqrt{x+2}-1}{x+1}$ is shown at right.

$$\lim_{x \to -1} \frac{\sqrt{x+2}-1}{x+1}$$



B. $\lim_{x \to 5} \frac{x-5}{3-\sqrt{x+4}}$

EX #6: Find each of the following limits analytically. Show your algebraic steps.

A. $\lim_{x \to 3} (5x+1)^{\frac{2}{3}}$

B. $\lim_{x \to 3} \frac{\sqrt{x+1}-2}{x-3}$

C. $\lim_{x \to -3^-} \frac{x^2|2x+6|}{4x+12}$

D. $\lim_{x \to 3^+} \frac{4x-12}{\sqrt{x^2-6x+9}}$

E. $\lim_{x \to 0} \frac{\frac{1}{x}+\frac{1}{x+2}}{x}$

F. $\lim_{h \to 0} \frac{(x+h)^2-3(x+h)-(x^2-3x)}{h}$

Topic 1.8: Determining Limits Using the Squeeze Theorem

The Squeeze Theorem is a technique used to confirm the limit of a function by comparison with two other functions whose limits are known or easily computed. Consider some function $f(x)$ is "trapped between two functions" on an interval containing point c. Let f, g, and h be functions defined on the interval except possibly at c itself. Then for $x \neq c$, in the interval $f(x) \leq g(x) \leq h(x)$ and, also that $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$. Then $\lim_{x \to c} g(x) = L$

EX #7: The Squeeze Theorem

A. If $1 \leq f(x) \leq x^2 + 2x + 2$, find $\lim_{x \to -1} f(x)$

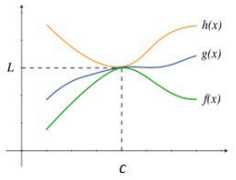
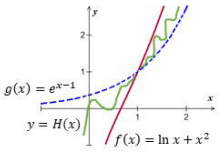


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B. Suppose $f(x)$ and $g(x)$ are boundaries of $H(x)$, as shown on the graph below. Given $f(x) = \ln x + x^2$ and $g(x) = e^{x-1}$, for all x in the interval containing $x = 1$, except possibly at $x = 1$ itself, find $\lim_{x \to 1} H(x)$. Justify.



NOTES

I.5 **Finding Limits by Analytic Methods**

Name _____

Date _____ Period _____

Problems 1 – 16, Find each of the following limits analytically. Show your algebraic analysis.

1. $\lim_{x\to 3}\left(\frac{2}{3}x^2+3x\right)$	**2.** $\lim_{t\to 4}\frac{t-4}{t^2-16}$
3. $\lim_{x\to -3}\frac{x^2-5x+6}{2x+6}$	**4.** $\lim_{\theta\to \pi}\sin^2\theta-3\cos\theta$
5. $\lim_{x\to 2}\frac{x^3-8}{x-2}$	**6.** $\lim_{\theta\to\frac{\pi}{3}}\frac{\tan\theta}{\theta^2}$
7. $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$	**8.** $\lim_{x\to 0}\frac{\frac{1}{3+x}-\frac{1}{3}}{x}$

9. $\lim_{x\to e}\frac{3x}{\ln x}$	**10.** $\lim_{x\to 2^+}\frac{3x^2+7x+2}{x^2-4}$
11. $\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}$	**12.** $\lim_{x\to\frac{3}{2}}\frac{8x^3-27}{2x-3}$
13. $\lim_{x\to\pi^+}\cot x$	**14.** $\lim_{x\to 0}\cos(x+\sin x)$
15. $f(x)=\begin{cases}3x-1, & x\le 1\\ 3x^2, & x>1\end{cases}$, find $\lim_{x\to 1}f(x)$	**16.** $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$

Problems 17 – 32, find the limits, if they exist, for the given piecewise-defined functions.

$$f(x)=\begin{cases}x+3, & x<2\\ x^2-1, & 2\le x<4\\ \sqrt{x+5}, & x\ge 4\end{cases}\qquad g(x)=\begin{cases}4-x^2, & x\le 1\\ 7, & 1<x<3\\ 2-x, & x\ge 3\end{cases}$$

17. $\lim_{x\to 2^+}f(x)$	**18.** $\lim_{x\to 2^-}f(x)$	**19.** $\lim_{x\to 2}f(x)$	**20.** $f(2)$
21. $\lim_{x\to 4^+}f(x)$	**22.** $\lim_{x\to 4^-}f(x)$	**23.** $\lim_{x\to 4}f(x)$	**24.** $f(4)$
25. $\lim_{x\to 1^-}g(x)$	**26.** $\lim_{x\to 1^+}g(x)$	**27.** $\lim_{x\to 1}g(x)$	**28.** $g(1)$
29. $\lim_{x\to 3^-}g(x)$	**30.** $\lim_{x\to 3^+}g(x)$	**31.** $\lim_{x\to 3}g(x)$	**32.** $g(3)$

33. If $2\le f(x)\le x^2+2$ for all x, find $\lim_{x\to 0}f(x)$. Sketch a graph to illustrate.

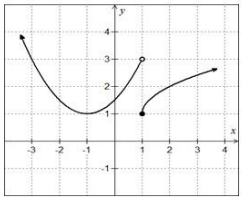
34. If $\lim_{x\to c}f(x)=-5$ and $\lim_{x\to c}g(x)=8$, find $\lim_{x\to c}\frac{2f(x)}{g(x)-f(x)}$

Practice Sets

1.5 Finding Limits Analytically Form A

Name _____

Date _____ Period _____



1. In the graph shown above, find the limits.

$\lim_{x \to 1^+} f(x) =$

$\lim_{x \to 1^-} f(x) =$

2. $\lim_{x \to -\infty} \left(\frac{3}{x+4}\right) =$

3. $\lim_{x \to 5} \left(\frac{7x}{x-5}\right) =$

4. $\lim_{x \to -3} \left(\frac{x^2+4x+3}{x^2-3}\right) =$

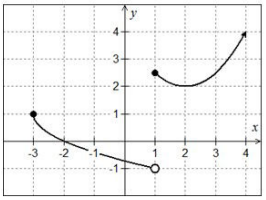
5. $\lim_{x \to 2} \sqrt{x+3} =$

Grade:

1.5 Finding Limits Analytically Form B

Name _____

Date _____ Period _____



1. In the graph shown above, find the limits.

$\lim_{x \to 1^+} f(x) =$

$\lim_{x \to 1^-} f(x) =$

2. $\lim_{x \to 2} \left(\frac{2x}{x-2}\right) =$

3. $\lim_{x \to \infty} \left(\frac{6}{x-3}\right) =$

4. $\lim_{x \to 0^-} \frac{|x-3|}{x} =$

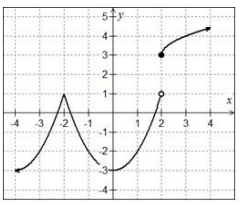
5. $\lim_{x \to -3} \left(\frac{x^2+5x+6}{x+3}\right) =$

Grade:

1.5 Finding Limits Analytically Form C

Name _____

Date _____ Period _____



1. In the graph shown above, find the limits.

$\lim_{x \to 2^+} f(x) =$

$\lim_{x \to 2^-} f(x) =$

2. $\lim_{x \to 8} \sqrt{2x-7} =$

3. $\lim_{x \to -4} \left(\frac{x+4}{x^2-16}\right) =$

4. $\lim_{x \to 2} \left(\frac{x^2-4x+4}{x^2-4}\right) =$

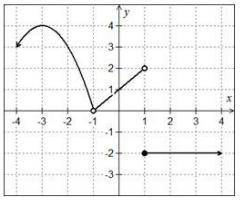
5. $\lim_{x \to 0^+} \frac{|x-4|}{x} =$

Grade:

1.5 Finding Limits Analytically Form D

Name _____

Date _____ Period _____



1. In the graph shown above, find the limits.

$\lim_{x \to 1^+} f(x) =$

$\lim_{x \to 1^-} f(x) =$

2. $\lim_{x \to 4} \left(\frac{6x}{x-4}\right) =$

3. $\lim_{x \to 5} \sqrt{9-x} =$

4. $\lim_{x \to 0^+} \frac{|x-2|}{x} =$

5. $\lim_{x \to 3} \left(\frac{x^2+6x-27}{x-3}\right) =$

Grade:

1.5 Finding Limits Analytically Form A Set 2

Name _____

Date _____ Period _____

1. $\lim_{x \to 2} \frac{x^2-x-2}{x-2}$

2. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$

3. $\lim_{x \to 0} \frac{\frac{1}{x+2}-\frac{1}{2}}{x}$

4. $\lim_{x \to 3} f(x)$ given $f(x) = \begin{cases} x-1, & x<3 \\ 2x-3, & x \geq 3 \end{cases}$

Grade:

1.5 Finding Limits Analytically Form B Set 2

Name _____

Date _____ Period _____

1. $\lim_{x \to -2} \frac{x^2-2x-8}{x+2}$

2. $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$

3. $\lim_{x \to 0} \frac{\frac{1}{3+x}-\frac{1}{3-x}}{x}$

4. $\lim_{x \to -1} f(x)$ given $f(x) = \begin{cases} x+4, & x \geq -1 \\ 3x-2, & x<-1 \end{cases}$

Grade:

1.5 Finding Limits Analytically Form C Set 2

Name _____

Date _____ Period _____

1. $\lim_{x \to 3} \frac{x^2+x-12}{x-3}$

2. $\lim_{x \to 16} \frac{x-16}{\sqrt{x}-4}$

3. $\lim_{x \to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$

4. $\lim_{x \to 2} f(x)$ given $f(x) = \begin{cases} x^2+1, & x>2 \\ 5(x-1), & x \leq 2 \end{cases}$

Grade:

1.5 Finding Limits Analytically Form D Set 2

Name _____

Date _____ Period _____

1. $\lim_{x \to -3} \frac{x^2-2x-15}{x+3}$

2. $\lim_{x \to 25} \frac{x-25}{\sqrt{x}-5}$

3. $\lim_{x \to 4} \frac{\frac{1}{x}-\frac{1}{4}}{x-4}$

4. $\lim_{x \to -2} f(x)$ given $f(x) = \begin{cases} x^2-9, & x \geq -2 \\ 4x+3, & x<-2 \end{cases}$

Grade:

Daily Lesson Quiz