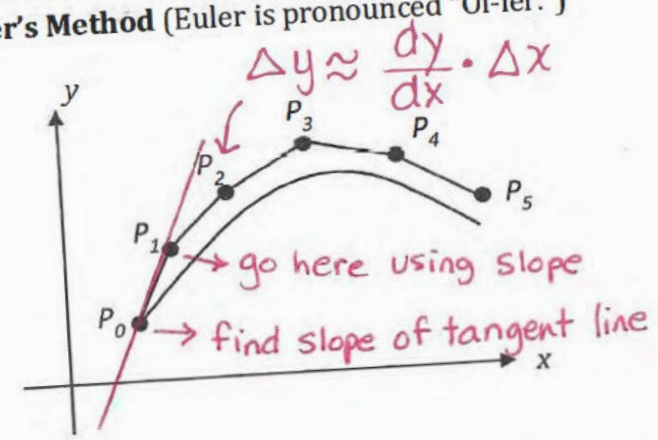
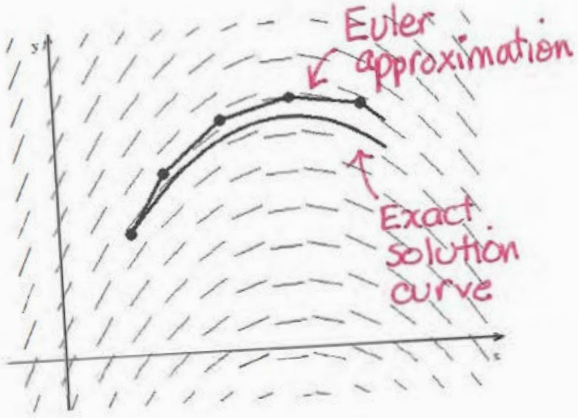
AP® Calculus BC

Euler's Method

7.3 Euler's Method

...dy of differential equations, we have only been given a restrictive set of linear, separable equations. However, there are many differential equations that cannot be solved for an explicit solution. We used a ***graphical approach*** in AB Calculus, where we used the slope field to sketch an approximate curve through a given point. Then, we "separated and integrated," for an ***algebraic method***. Now let's explore a ***numerical approach*** for approximating the solution of a differential equation, known as **Euler's Method** (Euler is pronounced "Oi-ler.")





The Big Idea of Euler's Method

(Leonhard Euler was an eighteenth-century Swiss mathematician)

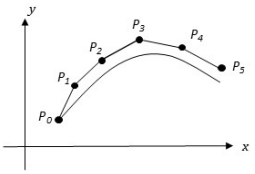
...display local linearity at any given point. When you have

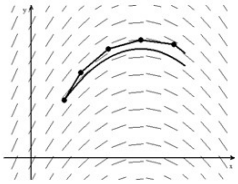
Completely Editable 7.3

Lesson 7.3: Euler's Method

Topic 7.5: Approximating Solutions Using Euler's Method

In our study of differential equations, we have only been given a restrictive set of linear, separable equations. However, there are many differential equations that cannot be solved for an explicit solution. We used a ***graphical approach*** in AB Calculus, where we used the slope field to sketch an approximate curve through a given point. Then, we "separated and integrated," for an ***algebraic method***. Now let's explore a ***numerical approach*** for approximating the solution of a differential equation, known as **Euler's Method** (Euler is pronounced "Oi-ler.")





The Big Idea of Euler's Method

(Leonhard Euler was an eighteenth-century Swiss mathematician)

Recall that a differentiable function will display local linearity at any given point. When you have zoomed in to a ***sufficiently small neighborhood*** around the point of tangency, it would be difficult to distinguish the function from its tangent line. This is the basis for Euler's Method. The slope field is our set of "signposts" on the plane for any given starting point (corresponding to the initial value). You can calculate the slope at that point using the differential equation. The slope tells you the direction to travel on a small segment. If you move a small amount and calculate a new slope from the differential equation, you can travel another small distance. Just repeat the process and you will approximate the actual curve.

Exploration: Write the equation of the tangent line for each point. Make a chart to keep you organized. Given: $\frac{dy}{dx} = y\,;\, y(0) = 1$ with $\Delta x = 0.1$

	x	y	Slope	Tangent line	$x + \Delta x$	New y
P_0						
P_1						
P_2						
P_3						
P_4						
P_5						

Euler's Method for Solving Differential Equations

- ❑ Solve the differential equation for dy in terms of x, y, and dx.
- ❑ Substitute values of x, y, and dx to calculate a value of dy. $[\Delta y = (slope)\Delta x]$.
- ❑ Find the approximate new value of y by adding dy to the previous value of y.
- ❑ Continue this procedure to find the next value of dy t the next value of x.

EX #1: Show that Euler's method for $\frac{dy}{dx} = 0.5(x + y)$ starting at $(0.2, 1.11)$ and using two steps with $\Delta x = -.05$ gives $y \approx 1.1775$ when $x = 0.3$.

	x	y	dy	New y
P_0	0.2	1.11		
P_1				
P_2				

EX #2: Approximate four points on the solution curve to $\frac{dy}{dx} = -3x^2y$, starting at $(0, 2)$ and using $\Delta x = 0.1$.

	x	$\Delta y = y'_{(previous)} \cdot \Delta x$	New y	(x, y)
P_0	0	----------	2	$(0, 2)$
P_1				
P_2				
P_3				
P_4				

The Accuracy of Euler's Method

The *error* in using Euler's Method is Exact Value – Approximate Value. If the number of steps used is n, the error is approximately proportional to $1/n$.

EX #3: Compare the tables below. Each represents the specific solution to the differential equation $\frac{dy}{dx} = y$ where $y(0) = 1$ with $\Delta x = 0.1$. Euler's method is an *approximation* of the particular solution.

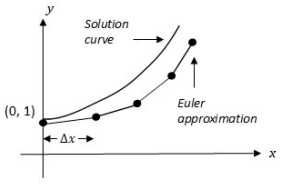
Euler's Approximation of $\frac{dy}{dx} = y$

x	y
0	1
0.1	1.1
0.2	1.21
0.3	1.331
0.4	1.4641
0.5	1.61051

Exact values of particular solution: $y = e^x$

x	y
0	1
0.1	1.10517...
0.2	1.2214...
0.3	1.3498...
0.4	1.4918...
0.5	1.6487...

The graph at right illustrates the particular solution as it compares to the Euler approximation.



A. Why do the approximate y-values underestimate the actual values?

B. Complete the table at right, comparing the difference between Exact Value — Approximate Value. Are the approximations getter better

x	Exact Value – Euler Approximation
0	

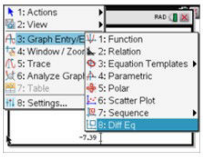
C. What can be changed to make the Euler solution curve more accurate?

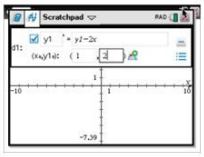
Technology Extension with TI-NSPIRE CX-CAS

EX #4: On the TI-NSPIRE CX-CAS, let's explore both the slope field and the approximate solution for $\frac{dy}{dx} = y - 2x$, at $(1, 2)$ with

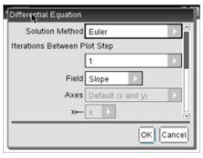
1. In the Scratchpad graph window, **Click 3 > 8**.

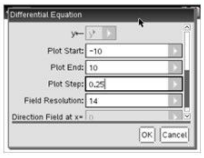
2. Enter the differential equation and the initial condition. Note that your slope field is entered as: $y1' = y1 - 2x$. **Click ENTER.**





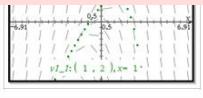
3. Next, edit parameters to meet the step of 0.25. **Click TAB**, then Up arrow to hover over the three horizontal dots at the right of the equation entry line. Notice the solution method is Euler. Scroll up to plot step and change the default to 0.25. **Click OK.**





4. Finally, **Click Menu > Trace > Graph Trace.**





NOTES

7.3 **Euler's Method Homework**

Name _____

Date _____ Period _____

Problems 1 - 4, Use Euler's Method to approximate the given value of dy/dx , with the given step size Δx.

1. Find $y(0.5)$

$\frac{dy}{dx} = 2y\,;\, y(0) = 3\,;\, \Delta x = 0.1$

x	$\Delta y = y'(previous)$	New y
0	----------	3

2. Find $y(3.25)$

$\frac{dy}{dx} = x^2 - y\,;\, y(3) = 1;\, \Delta x = 0.05$

x	$\Delta y = y'(previous)$	New y
3	----------	1

3. Find $y(2)$

$\frac{dy}{dx} = x \sin y\,;\, y(1) = 3\,;\, \Delta x = 0.2$

x	$\Delta y = y'(previous)$	New y
1	----------	3

4. Find $y(1)$

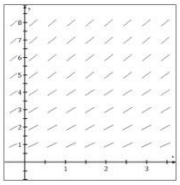
$\frac{dy}{dx} = \cos x + \sin y\,;\, y(0) = 4\,;\, \Delta x = 0.25$

x	$\Delta y = y'(previous)$	New y
0	----------	4

Problems 5-6, Use Euler's Method with the given step size Δx to approximate the solution of the initial-value problem over the stated interval. Sketch the graph of your approximation.

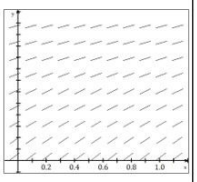
5. $\frac{dy}{dx} = \sqrt[3]{y}\,;\; y(0) = 2; 0 \le x \le 4\,;\, \Delta x = 0.5$

x	y



6. $\frac{dy}{dt} = e^{-y}\,;\, y(0) = 1\,;\, 0 \le t \le 1, \Delta t = 0.1$

x	y



7. Show that Euler's method for $\frac{dy}{dx} = y + \cos x$ with initial condition $(0, 0)$ and using three steps with $\Delta x = 0.1$ gives $y(0.3) \approx 0.3285$

8. Approximate four points on the solution curve to $\frac{dy}{dx} = \frac{2x}{y}$ starting at $(0, 2)$ and using $\Delta x = 0.25$

9. Let $y = g(x)$ be a solution to the differential equation $\frac{dy}{dx} = 3x - 4y$ with initial condition $g(0) = k$, where k is a constant. Euler's method starting at $x = 0$, with a step size of 0.5 approximates $g(2) = 2.25$. Find the value of k.

10. Consider the initial-value problem $\frac{dy}{dx} = -3x^2y\,;\, y(0) = 2$.

A. Use Euler's method with step sizes of $\Delta x = 0.2$ to find an approximation for $y(1)$. Sketch the solution on the slope field below.

B. Solve the differential equation to find the value for $y(1)$.

x	$\Delta y = y'(previous) \cdot \Delta x$	New y
0	----------	2

C. Compare the two values. Does your estimate in A

D. Find the error of your approximation.

11. A large tanker ship is carrying oil barrels to port. There is a large barrel in the hull that is leaking oil at an increasing rate of $V'(t) = 0.5\,(1 + t)^2$.



A. Estimate the amount of oil that will leak into the hull in 5 hours, using increments of 1 hour. At $t = 0$, the barrel has already leaked 30 gallons.

x	$\Delta y = y'(previous) \cdot \Delta x$	New y
0	----------	30

B. Solve the differential equation algebraically.

D. Find the error of your Euler approximation.

Practice Sets

7.3 Euler's Method **Form A**

Name _____

Date _____ Period _____

1. Find $y(1)$

$\frac{dy}{dx} = \frac{\sqrt{y}}{2}$; $y(0) = 1$; $\Delta x = 0.2$

x	$\Delta y = y'(\text{previous}) \cdot \Delta x$	New y
0	----------	1

2. Find the particular solution to $\frac{dy}{dx} = \frac{\sqrt{y}}{2}$ and calculate $y(1)$.

7.3 Euler's Method **Form B**

Name _____

Date _____ Period _____

1. Find $y(1)$

$\frac{dy}{dx} = 2y$; $y(0) = 2$; $\Delta x = 0.2$

x	$\Delta y = y'(\text{previous}) \cdot \Delta x$	New y
0	----------	2

2. Find the particular solution to $\frac{dy}{dx} = 2y$ and calculate $y(1)$.

7.3 Euler's Method **Form C**

Name _____

Date _____ Period _____

1. Find $y(1)$

$\frac{dy}{dx} = x + 2$; $y(0) = 3$; $\Delta x = 0.2$

x	$\Delta y = y'(\text{previous}) \cdot \Delta x$	New y
0	----------	1

2. Find the particular solution to $\frac{dy}{dx} = x + 2$ and calculate $y(1)$.

7.3 Euler's Method **Form D**

Name _____

Date _____ Period _____

1. Find $y(1)$

$\frac{dy}{dx} = x - 3$; $y(0) = 5$; $\Delta x = 0.2$

x	$\Delta y = y'(\text{previous}) \cdot \Delta x$	New y
0	----------	2

2. Find the particular solution to $\frac{dy}{dx} = x - 3$ and calculate $y(1)$.

Daily Lesson Quiz