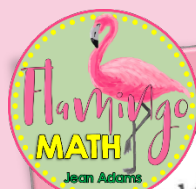


AP[®] Calculus

Differentiation-Part 2

Composite, Implicit, and Inverse Functions



Given $x^2 - xy + y^2 = 7$

A. Find the equation of the tangent line at $x = (-1, 2)$.

$$2x - x \cdot \frac{dy}{dx} - 1y + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}[2y - x] = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

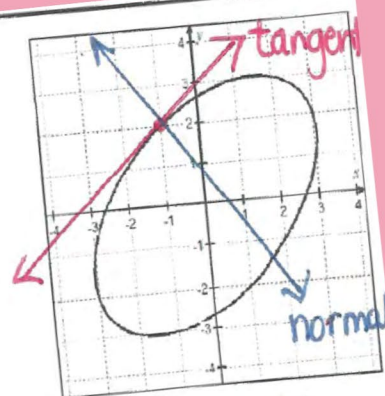
B. Find the equation of the normal line at $x = (-1, 2)$

Tangent Line: $y = \frac{4}{5}(x+1) + 2$

Normal Line: $y = -\frac{5}{4}(x+1) + 2$

Find:

$$y' \Big|_{(-1, 2)} = \frac{2+2}{4+1} = \frac{4}{5}$$



$$m_{\perp} = -\frac{5}{4}$$

Guided Notes

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Lesson I: The Chain Rule

Topic 3.1: The Chain Rule

The chain rule is used to differentiate composite functions such as $y = \sqrt{x^3 + 2}$ or $y = \sin(x^2)$. Remember that you create a composite function by “plugging” one function into another function. The composition of f and g is denoted by $(f \circ g)(x) = f(g(x))$. We can find the derivative of each separate function by using the rules we’ve learned so far. But, we need the Chain Rule to find the derivative of two functions “plugged into one another.”

The Chain Rule:

If f and g are differentiable, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable and $f'(g(x)) \cdot g'(x)$

Leibniz notation will allow us to use “ u -substitution.” Let $y = f(u)$ and $u = g(x)$, notice that y is a function of u and u is a function of x . So it follows that,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This looks more complicated than it really is. Basically, the Chain Rule says to:

MULTIPLY THE DERIVATIVE OF THE INSIDE FUNCTION BY THE DERIVATIVE OF THE OUTSIDE FUNCTION.

EX #1: Find the derivative of $f(x) = (5x + 3)^2$ with and without the chain rule.

A. Without the chain rule

B. With the chain rule

EX #2: Find $f'(x)$ given $f(x) = (4 - x^2)^3$

$f'(x) = (\text{Derivative of Outside Function}) \times (\text{Derivative of Inside Function})$

Lesson 2: Symbolic Differentiation

Topic 3.5: Selecting Procedures for Calculating Derivatives

There will be times when you need to find an approximation of the derivative. Typically, this type of question will be given in the form of a table of values without a specific function. You will need to identify an appropriate mathematical rule or procedure based on the given data to proceed successfully.

EX #1: The table gives the values of a function that is differentiable on the interval $[0, 1]$. Find $f'(0.10)$.

x	0.10	0.20	0.30	0.40	0.50
$f(x)$	0.261	0.433	0.552	0.587	0.534

EX #2: Symbolic Differentiation

Given the table of values below, find each of the derivatives of the following at $x = 2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	-4	8	3	-1
3	5	-2	1	-3
4	0	-1	2	3

A. $H(x) = f(g(x))$

B. $K(x) = [f(x)]^3$

C. $P(x) = f(x) \cdot g(2x)$

D. $R(x) = \frac{f(2x)}{g(x)}$

Lesson 4: Derivatives of Inverse Functions

Topic 3.3: Differentiating Inverse Functions

Recall from previous courses that a function, $y = f(x)$, that is **one-to-one** will pass the **horizontal line test** and will therefore, have a unique **inverse function** $y = f^{-1}(x)$. We often say that the inverse function “undoes” the original function $f(x)$. This can be seen in the test to prove two functions are inverses of each other, since the composition of the two functions produces the identity function.

$$f(f^{-1}(x)) = x = f^{-1}(f(x))$$

Definition of an Inverse:

A function g is the inverse function of the function f if $f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f . The function $g(x)$ is denoted by $f^{-1}(x)$.

Reflective Property of Inverse Functions:

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a) .

EX #1: Use the reflective property of inverse functions and the function values for $f(x)$ and $g(x)$ in the table at right, to complete the table of values for their respective inverse functions, $f^{-1}(x)$ and $g^{-1}(x)$. Then, use them to answer each of the questions below.

x	$f(x)$	$g(x)$
3	2	-1
1	-1	0
-1	4	3

A. The table of values for $f^{-1}(x)$:

x	$f^{-1}(x)$

B. The table of values for $g^{-1}(x)$:

x	$g^{-1}(x)$

Use the tables above to evaluate each of the following expressions below:

C. $f(g^{-1}(0))$

D. $f^{-1}(f(-1))$

E. $f^{-1}(g^{-1}(-1))$

F. $g^{-1}(f^{-1}(4))$

To Find the Derivative of an Inverse Function:

Let f and g be inverse functions, such that $f(g(x)) = x = g(f(x))$ where $f(a) = b$ and $g(b) = a$.

To find $g'(b)$ for a point (a, b) on $f(x)$.

1. Find $f'(x)$
2. If you are only given b , set $b = f(x)$ to find a .
3. Find $f'(a)$
4. $g'(b) = \frac{1}{f'(a)}$

NOTE: $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

***** Inverse functions have reciprocal slopes at corresponding points. *****

EX #3: Find $g'(-1)$; where $g(x)$ is the inverse of $f(x)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	5	0		
3	-1	-4		
-1	4	6		

EX #4: Let g be the inverse of f . Calculate $g'(-9)$ for $f(x) = 4x^3 + 5x$

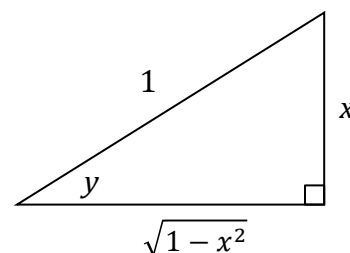
Lesson 5: Derivatives of Inverse Trigonometric Functions

Topic 3.4: Differentiating Inverse Trigonometric Functions

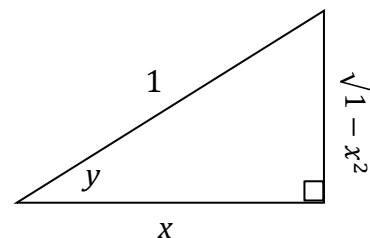
In this final lesson for the unit we can take the derivative of an inverse trigonometric function using the ratios of a right triangle and implicit differentiation. We will start with sine inverse. Let's derive!

EX #1: Use the Pythagorean Theorem, properties of right triangle trigonometry and implicit differentiation to derive the inverse trigonometric function formulas.

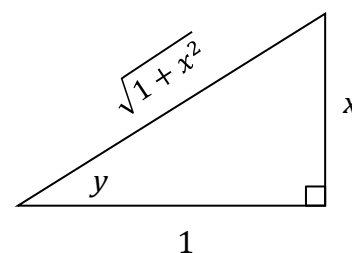
A. Given $y = \sin^{-1}(x)$, find $\frac{d}{dx}(\sin^{-1} x)$



B. Given $y = \cos^{-1}(x)$, find $\frac{d}{dx}(\cos^{-1} x)$



C. Given $y = \tan^{-1}(x)$, find $\frac{d}{dx}(\tan^{-1} x)$



Derivatives of Inverse Trigonometric Functions

NOTEWORTHY: On previous exams, it is most important to know the inverse formulas for sine, cosine, and tangent. You can memorize them or learn to derive them!

$$\frac{d}{dx} [\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} [\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx} [\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} [\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx} [\arctan u] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx} [\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

EX #2: Differentiate.

A. $y = \sin^{-1}(5x)$

B. $y = \tan^{-1}(\sqrt{2x+1})$

C. $y = \arccos(x^2)$

D. $y = \sin^{-1}(x^2 - 1)$

E. $y = \tan^{-1}(\cos x)$

F. $y = \sin^{-1}(e^x)$