

AP[®] Calculus-BC

Differential Equations



- A. Population of coyotes in a wildlife preserve is modeled by a logistic function. The preserve can support no more than 100 coyotes on the property. Write the differential equation with $k = 0.1$, where t is measured in years.

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{100}\right)$$
$$= 0.001P(100 - P)$$

- B. The slope field is shown, at right. Where does the horizontal asymptote appear to occur?

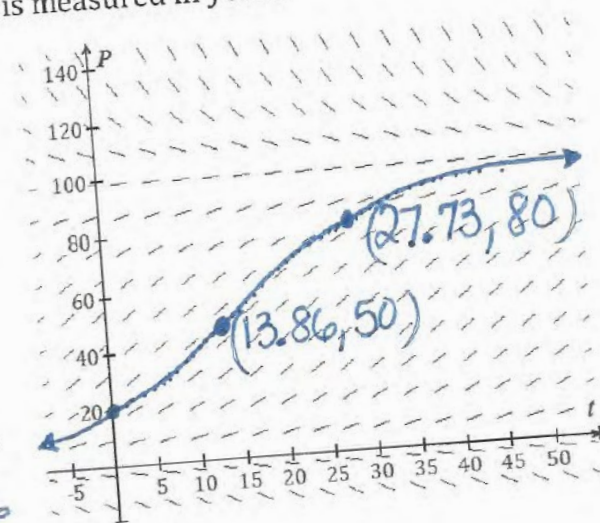
$$y = 100$$

carrying capacity

- C. What happens if the initial population is above this asymptote?

The population is in decline overpopulated for the preserve

- D. If the preserve initially begins with twenty coyotes, sketch a graph of $P(t)$ on the slope field.



Guided Notes

7

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AP CALCULUS BC CURRICULUM

Scope and Sequence

Unit 7: Differential Equations

Lesson	Skills & Learning Objectives
[7.1] Introduction to Differential Equations	☐ Interpret verbal statements to write differential equations and verify solutions.
[7.2] Slope Fields	☐ Use slope fields to approximate solutions of differential equations. ☐ Identify slope fields for differential equations ☐ Use initial conditions to find particular solutions of differential equations.
[7.3] Euler's Method	☐ Use a differential equation and its slope field to calculate points on the graph iteratively using Euler's Method.
[7.4] Separation of Variables	☐ Use separation of variables to solve simple differential equations.
[7.5] Growth and Decay	☐ Use exponential functions to model growth and decay in applied problems.
[7.6] Logistic Growth	☐ Use logistic differential equations to model and solve applied problems including Newton's Cooling Law.

Lesson 7.1: Introduction to Differential Equations

Topic 7.1: Modeling Situations with Differential Equations

Before studying calculus, when we solved equations containing numbers and variables, our solutions were numbers. Now we will explore differential equations where our solution will be a function.

A **differential equation** is any equation that involves one or more derivatives of a function. These equations describe relationships that involve rates of change.

Consider the following statement:

"The population, P , of your state is growing at an instantaneous rate of 3% of the population per year."

The derivative of the population $\frac{dP}{dt} = 0.03P$ is an example of a **differential equation**.

EX #1: Write a differential equation that describes each relationship stated below.

A. Alex and Tina have a new Maine Coon kitten named *King Robert*. Robert is gaining weight, w at a rate that is inversely proportional to his age, t , in months.

B. Your cup of Starbite's Peppermint Coffee is cooling at a rate that is proportional to the difference between the room temperature T_R and the current temperature, T of the coffee.

Topic 7.2: Verifying Solutions for Differential Equations

The **general solution** of a first-order differential equation represents a **family of curves**, which we will investigate in this unit. First let's look at some of the notation methods for representing differential equations.

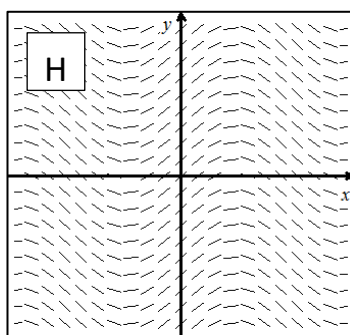
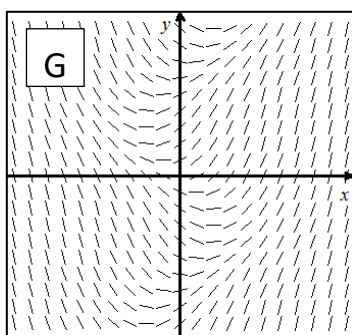
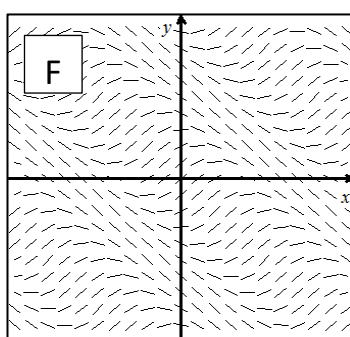
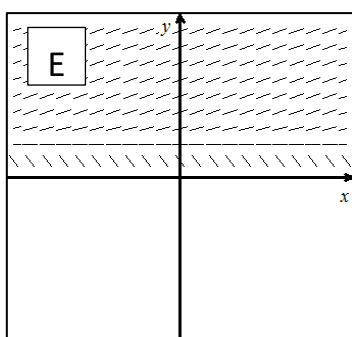
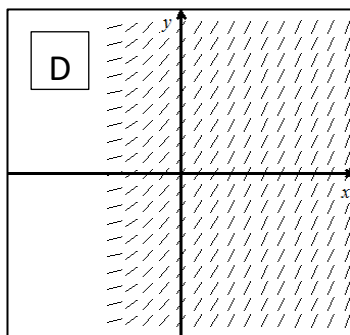
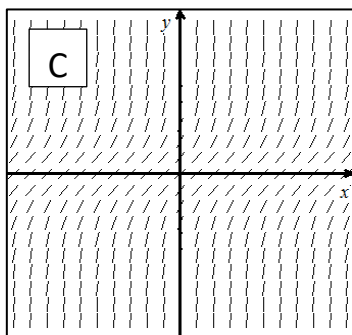
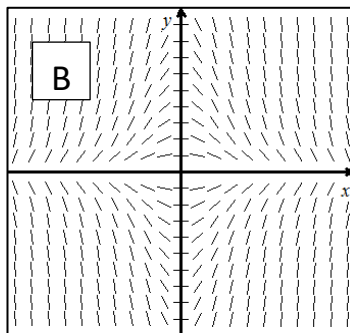
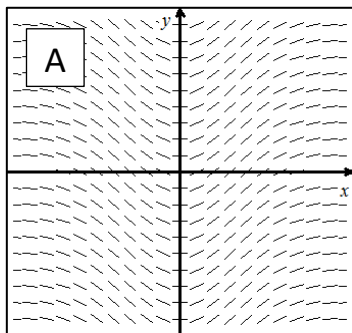
Multiple Representations of Differential Equations	
$3y' + y'' = 4y$	Differential Equation
$3f'(x) + f''(x) = 4f(x)$	Function notation
$3 \frac{dy}{dx} + \frac{d^2y}{dx^2} = 4f(x)$	Liebniz notation

To see how there is a whole "*class of functions*" that can satisfy this particular differential equation, consider the function family of $y = e^{bx}$. In our next example, we will show how there are many functions that can satisfy this differential equation.

G. Explain what happens to the slope values as you move from left to right with a fixed y value.

H. Explain what happens to the slope values as you move up from the x -axis with a fixed x value.

EX #4: Matching



_____ 1.) $\frac{dy}{dx} = \sqrt{x+2}$

_____ 2.) $\frac{dy}{dx} = xe^{-0.2x^2}$

_____ 3.) $\frac{dy}{dx} = \sin(y) + x$

_____ 4.) $\frac{dy}{dx} = \cos x$

_____ 5.) $\frac{dy}{dx} = \frac{\ln y}{y}$

_____ 6.) $\frac{dy}{dx} = y^2 + 1$

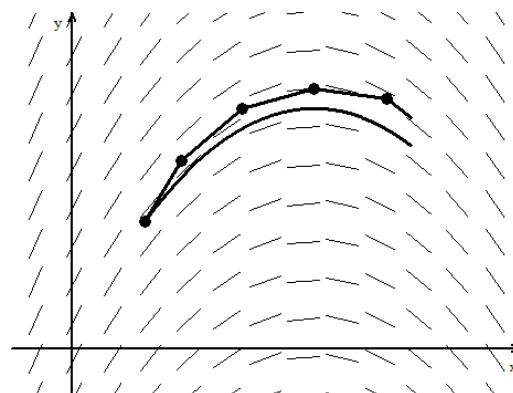
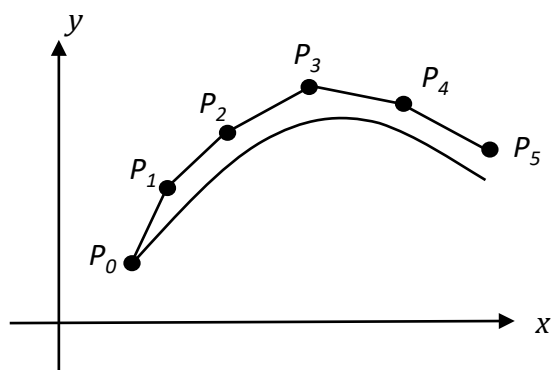
_____ 7.) $\frac{dy}{dx} = -xy$

_____ 8.) $\frac{dy}{dx} = \cos(x+y)$

Lesson 7.3: Euler's Method

Topic 7.5: Approximating Solutions Using Euler's Method

In our study of differential equations, we have only been given a restrictive set of linear, separable equations. However, there are many differential equations that cannot be solved for an explicit solution. We used a **graphical approach** in AB Calculus, where we used the slope field to sketch an approximate curve through a given point. Then, we “separated and integrated,” for an **algebraic method**. Now let's explore a **numerical approach** for approximating the solution of a differential equation, known as **Euler's Method** (Euler is pronounced “Oi-ler.”)



The Big Idea of Euler's Method

(Leonhard Euler was an eighteenth-century Swiss mathematician)

Recall that a differentiable function will display local linearity at any given point. When you have zoomed in to a **sufficiently small neighborhood** around the point of tangency, it would be difficult to distinguish the function from its tangent line. This is the basis for Euler's Method. The slope field is our set of “signposts” on the plane for any given starting point (corresponding to the initial value). You can calculate the slope at that point using the differential equation. The slope tells you the direction to travel on a small segment. If you move a small amount and calculate a new slope from the differential equation, you can travel another small distance. Just repeat the process and you will approximate the actual curve.

Exploration: Write the equation of the tangent line for each point. Make a chart to keep you organized. Given: $\frac{dy}{dx} = y$; $y(0) = 1$ with $\Delta x = 0.1$

	x	y	Slope	Tangent line	$x + \Delta x$	New y
P_0						
P_1						
P_2						
P_3						
P_4						
P_5						

The Accuracy of Euler's Method

The *error* in using Euler's Method is Exact Value – Approximate Value. If the number of steps used is n , the error is approximately proportional to $1/n$.

EX #3: Compare the tables below. Each represents the specific solution to the differential equation $\frac{dy}{dx} = y$ where $y(0) = 1$ with $\Delta x = 0.1$. Euler's method is an *approximation* of the particular solution.

Euler's Approximation of $\frac{dy}{dx} = y$

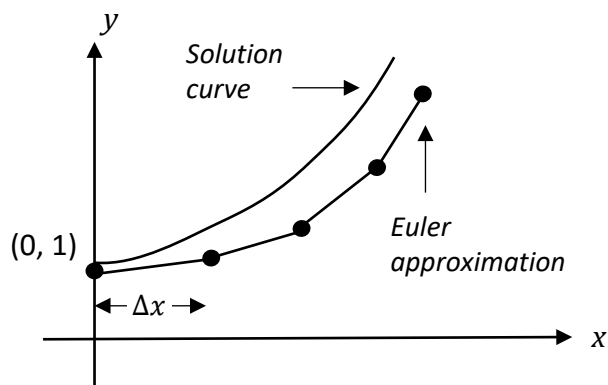
x	y
0	1
0.1	1.1
0.2	1.21
0.3	1.331
0.4	1.4641
0.5	1.61051

Exact values of particular solution: $y = e^x$

x	y
0	1
0.1	1.10517...
0.2	1.2214...
0.3	1.3498...
0.4	1.4918...
0.5	1.6487...

The graph at right illustrates the particular solution as it compares to the Euler approximation.

A. Why do the approximate y -values underestimate the actual values?



B. Complete the table at right, comparing the difference between Exact Value — Approximate Value. Are the approximations getting better or worse? Why?

x	Exact Value – Euler Approximation
0	
0.1	
0.2	
0.3	
0.4	
0.5	

C. What can be changed to make the Euler solution curve more accurate?

Lesson 7.6: Logistic Growth and Newton's Cooling Law

Topic 7.9: Logistic Models with Differential Equations

There are four types of differential equations for which we are responsible in BC Calculus. The first two types were covered in AB Calculus.

Type:	Form:	Solution:
Type 1: Position, Velocity, Acceleration	$a(t) = \frac{dv}{dt} = -g$	$v(t) = -gt + C$
Type 2: Simple Growth and Decay	$\frac{dy}{dt} = ky$	$y = Ce^{kt}$

The differential equation for exponential growth presented an assumption that growth continued forever. However, we understand that the real world does not behave in this manner. For example, when a medical epidemic breaks out, we know that the effects will be limited to a designated population. This “number of the population” is known as the **carrying capacity** or **limiting factor**. Most real-world situations do not grow without bound and can be modeled by the logistic differential equation.

Type:	Form:	Solution:
Type 3: Logistic Differential Equation	$\frac{dP}{dt} = kP \left(1 - \frac{P}{L}\right)$	$P = \frac{L}{1 + Ae^{-kt}}$

EX #1: Solve the logistic differential equation, where k and L are positive constants:

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{L}\right)$$

Procedure:

1. Separate variables
2. Integrate both sides
3. Use partial fractions to rewrite left side
4. Find antiderivative of each side
5. Multiply both sides by -1
6. Simplify
7. Exponentiate
8. Let $A = \pm e^{-c}$
9. Solve for y

NOTE:

The parameter L represents the limiting value. The parameter k represents the relative growth rate when P is small relative to L. The parameter A depends on P_0 .

$$A = \frac{L - P_0}{P_0}$$

Type:	Form:	Solution:
Type 4: Newton's Law of Cooling	$\frac{dy}{dt} = k(y - B)$ Growth $k > 0$ Decay $k < 0$	$y = Ae^{kt} + B$ Where $A = y_0 - B$ and y_0 is the initial quantity
Important:	Factor out the k before you start solving the differential equation!	

Typical applications of this type are as follows:

- ☐ Newton's Law of Heating and Cooling
- ☐ Falling objects approaching terminal velocity
- ☐ Pressure inside deflating containers approaching outside pressure

EX #3: You bought your favorite frozen coffee treat, on the way to school, at the drive-through window of **StarBeams**. Your car is always kept at a constant 65°F. Traffic was terrible, and you spent 10 minutes getting to school. The drink warmed to 35°F because you checked it with your temperature probe at the red light. Then, two students had a rear end collision and another 10 minutes passed trying to get into the school parking lot. You checked your drink again, it warmed to 50°F. Use Newton's Law of Cooling to estimate the following:

A. What was the frozen coffee's original temperature?