

AP[®] Calculus

Differentiability and Continuity

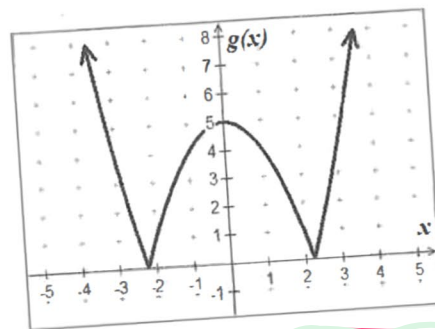


The contrapositive to the statement, "If a function is differentiable at $x=c$, then the function is continuous at $x=c$."

If $f(x)$ is NOT continuous at $x=c$, then
 $f(x)$ is NOT differentiable at $x=c$

EX #4: Describe the x -values at which the function $g(x) = |x^2 - 5|$ is differentiable.
Justify your answer.

$x^2 - 5 = 0$
 $x = \pm\sqrt{5}$
 $(-\infty, -\sqrt{5}) \cup (-\sqrt{5}, \sqrt{5}) \cup (\sqrt{5}, \infty)$
Since $g(x)$ has a sharp corner at $x = -\sqrt{5}$ and $x = \sqrt{5}$



Completely Editable

2.4

Lesson 4: Differentiability and Continuity

Topic 2.4: Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist

In this lesson we will investigate the relationship between differentiability and continuity. Our focus will be to determine when a function fails to have a derivative. A function that is differentiable at a point is also **locally linear at that point**. Basically, this is telling us that in order for a function to be differentiable at a point, the graph must be continuous and smooth at the point. Remember, when we say a function is differentiable, this means the derivative exists or you are able to take the derivative.

Differentiability at a Point

A function $y = f(x)$ is differentiable at $x = c$ if and only if

$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} = L = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}$$

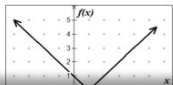
Where L is a finite value.

How $f'(a)$ May Fail to Exist . . .

The following examples will show us how a function will fail to have a derivative at a point where there is a **corner**, a **cusp**, a **vertical tangent line** or a **discontinuity**.

EX #1: Given the graph of $f(x) = |x - 1|$

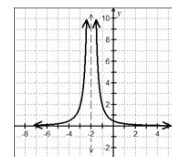
- What is $f'(x)$ as $x \rightarrow 1^-$
- What is $f'(x)$ as $x \rightarrow 1^+$
- Is f continuous at $x = 1$?
- Is f differentiable at $x = 1$?



EX #2: Given the graph of $f(x) = \sqrt[3]{x}$, use the alternate derivative $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ to show that there is a vertical tangent line at $x = 0$.

EX #3: Given the graph of $f(x) = \frac{1}{(x+2)^2}$

- A. Describe the derivative of f as x approaches negative two from the left and the right.



- B. Given the derivative $f'(x) = \frac{-2}{(x+2)^3}$, use the formula to evaluate the derivative when $x = -2$

Differentiability Implies Continuity:

If f is differentiable at $x = c$, then f is continuous at $x = c$.

- If a function is differentiable at $x = c$, then it is continuous at $x = c$.

So, **differentiability implies continuity**.

- It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$.

Continuity does not imply differentiability.

SHARP CORNER, CUSP, VERTICAL TANGENT LINE or DISCONTINUITY

Write the contrapositive to the statement, "If a function is differentiable at $x=c$, then the function is continuous at $x=c$."

EX #5: Find all values of x for which the function is differentiable.

$$h(x) = \begin{cases} (x+1)^2, & x \leq 0 \\ 2x+1, & 0 < x < 3 \\ (4-x)^2, & x \geq 3 \end{cases}$$

Piecewise Functions are always challenging. Not only do the limits of the slopes need to be approaching the same value on either side (piece) of a given x -value, but the function must be **connected and smooth** in order for the function to be differentiable at the given point.

EX #6: Find the value(s) of the constant a where the function is differentiable.

$$f(x) = \begin{cases} ax^2 - x, & x < 3 \\ 2a, & x \geq 3 \end{cases}$$

EX #9: Use your calculator to find the derivative of our first three functions in this lesson set. Explain any problems* that you encounter.

A. $f(x) = |x - 1|$ at $x = 1$

B. $f(x) = \sqrt[3]{x}$ at $x = 0$

C. $f(x) = \frac{1}{(x+2)^2}$ at $x = -2$

* Your calculator is programmed to find derivatives by using the symmetric difference quotient. This can cause errors. YOU MUST MAKE THE FINAL DECISION BASED ON PROPER ANALYSIS!

EX #10: Defending your answers. If you are given that $y = f(x)$ is a differentiable function at $x = 3$, explain why each statement below is true.

- $f(3)$ exists
- $\lim_{x \rightarrow 3} f(x)$ exists

Topic 2.3: Estimating Derivatives of a Function

On the AP Exam you will need to find the numeric derivative of a function at a given point. You can use the **nDeriv** feature on the TI-83+ or TI-84+ and the Derivative at a point feature on the TI-Nspire CX.

TI-83+ or TI-84+ Keystrokes

MATH>8
Display
Enter: dv

$$\frac{d}{dx} (x^2 - 5x + 6) \bigg|_{x=3} = -1$$

TI-Nspire CX or CX-CAS Keystrokes

MENU>4>2
Enter: an x-value
Enter: function



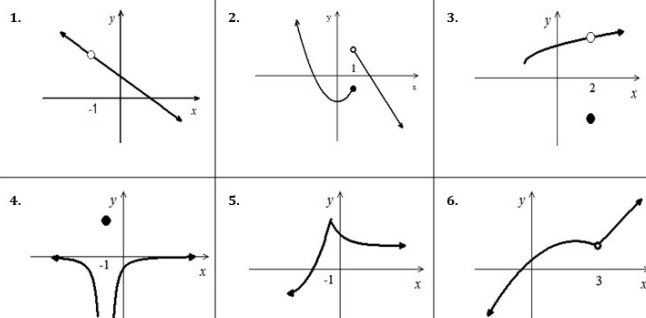
NOTES

Express your answer using proper notation.



D. $\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$ exists

Problems 1-6, determine if the function $f(x)$ is continuous at the value c . If not continuous, state which rule of continuity the function fails.



Problems 7-10, state the value(s) of x where the function is discontinuous. Justify your reasoning.

7. $f(x) = \frac{x^2 + 5x - 24}{x^2 - 64}$

8. $g(x) = \tan x$

9. $h(x) = 2^x + x^2$

10. $f(x) = \frac{x^3 - 27}{x - 3}$

Problems 11-12, determine whether the function is continuous at the "transition value," where the function rules change, by using the definition of continuity.

11. $f(x) = \begin{cases} x^2 - 3, & x < 1 \\ 2 - x, & x \geq 1 \end{cases}$

12. $g(x) = \begin{cases} 2(x + 2), & x \leq -1 \\ \sqrt{x + 1} + 2, & x > -1 \end{cases}$

Problems 13-16, find the value of the constant k that makes the function continuous. Show your set up and work.

13. $f(x) = \begin{cases} x^2 + 1, & x < 2 \\ kx - 3, & x \geq 2 \end{cases}$

14. $f(x) = \begin{cases} \sqrt{2x}, & x > 2 \\ kx + 1, & x \leq 2 \end{cases}$

15. $f(x) = \begin{cases} kx + 5, & x < -1 \\ kx^2, & x \geq -1 \end{cases}$

16. $f(x) = \begin{cases} x^2 + 3, & x \geq -2 \\ kx^2 - 2x, & x < -2 \end{cases}$

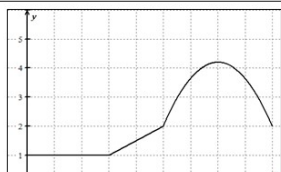
Problems 17 - 18, Use the alternate form of the derivative to determine whether each function is differentiable at the given x -value. Then, verify that the function is continuous at $x = c$.

17. $f(x) = \frac{1}{x}$ at $x = 2$

18. $f(x) = x^3 + 3x^2 + 4$ at $x = -1$

Problems 19 - 23: Use the graph of the function $f(x)$ shown in the figure at right.

19. $f'(4) =$ _____



20. For which values of x is $f'(x) < 0$?

Multiple Choice.

24. If $\lim_{x \rightarrow 3} f(x) = 9$, which of the following must be true?

- I. f is continuous at $x = 3$
- II. f is differentiable at $x = 3$
- III. $f(3) = 9$

- | | | | |
|---------|------------|-------------------|-------------------|
| A. None | B. II Only | C. I and III Only | D. I, II, and III |
|---------|------------|-------------------|-------------------|

25. At $x = 2$, the function given by $g(x) = \begin{cases} 2x, & x < 2 \\ x^2, & x \geq 2 \end{cases}$ is

- A. Differentiable but not continuous
- B. Continuous but not differentiable
- C. Both continuous and differentiable
- D. Neither continuous nor differentiable

Problems 26 - 28: Evaluate each of the following on your calculator. Explain any incorrect answers.



26. Find $\left. \frac{dy}{dx} \right|_{x=2}$ given $y = \ln x$

28. $y(1) = y - \sqrt{x - 1}$

Practice Sets

23. Which is larger $f'(5.5)$ or $f'(6.5)$?

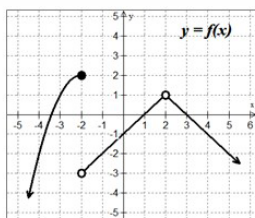
2.4 Differentiability & Continuity Form A

Name _____

Date _____ Period _____

1. Find the value of the constant k that makes the function continuous.

$$f(x) = \begin{cases} x^2, & x > 3 \\ k - x, & x \leq 3 \end{cases}$$



2. In the graph shown above, use the three rules of continuity to discuss where the function is not continuous or not differentiable.

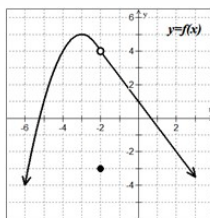
2.4 Differentiability & Continuity Form B

Name _____

Date _____ Period _____

1. Find the value of the constant k that makes the function continuous.

$$f(x) = \begin{cases} kx + 1, & x > 2 \\ 4 - x^2, & x \leq 2 \end{cases}$$



2. In the graph shown above, use the three rules of continuity to discuss where the function is not continuous or not differentiable.

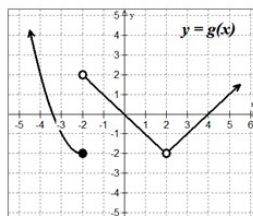
2.4 Differentiability & Continuity Form C

Name _____

Date _____ Period _____

1. Find the value of the constant k that makes the function continuous.

$$f(x) = \begin{cases} kx^2, & x > -1 \\ kx + 7, & x \leq -1 \end{cases}$$



2. In the graph shown above, use the three rules of continuity to discuss where the function is not continuous or not differentiable.

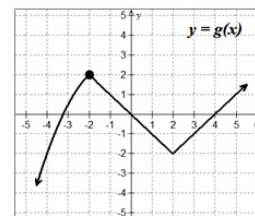
2.4 Differentiability & Continuity Form D

Name _____

Date _____ Period _____

1. Find the value of the constant k that makes the function continuous.

$$f(x) = \begin{cases} 1.5kx, & x < 2 \\ k^2 - x^2, & x \geq 2 \end{cases}$$



2. In the graph shown above, use the three rules of continuity to discuss where the function is not continuous or not differentiable.

Daily Lesson Quiz