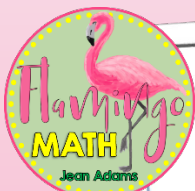


AP<sup>®</sup> Calculus

# Derivatives of Inverse Functions



relationship between a composite function and its inverse is stated below. If you use the chain and differentiate both sides, you can find a formula for the derivative of an inverse.

$$f[f^{-1}(x)] = x$$

Proof!

① Let  $g = f^{-1}(x)$

$$f(g(x)) = x$$
$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[x]$$

② By chain rule:

$$f'(g(x)) \cdot g'(x) = \frac{dx}{dx}$$

$$\frac{dx}{dx} = 1 \quad \text{③}$$

$$f'(g(x)) \cdot g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))}$$

④ By substitution from given

$$f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Completely Editable

3.4

## Lesson 4: Derivatives of Inverse Functions

### Topic 3.3: Differentiating Inverse Functions

Recall from previous courses that a function,  $y = f(x)$ , that is **one-to-one** will pass the **horizontal line test** and will therefore, have a unique **inverse function**  $y = f^{-1}(x)$ . We often say that the inverse function "undoes" the original function  $f(x)$ . This can be seen in the test to prove two functions are inverses of each other, since the composition of the two functions produces the identity function.

$$f(f^{-1}(x)) = x = f^{-1}(f(x))$$

#### Definition of an Inverse:

A function  $g$  is the inverse function of the function  $f$  if  $f(g(x)) = x$  for each  $x$  in the domain of  $g$  and  $g(f(x)) = x$  for each  $x$  in the domain of  $f$ . The function  $g(x)$  is denoted by  $f^{-1}(x)$ .

#### Reflective Property of Inverse Functions:

The graph of  $f$  contains the point  $(a, b)$  if and only if the graph of  $f^{-1}$  contains the point  $(b, a)$ .

**EX #1:** Use the reflective property of inverse functions and the function values for  $f(x)$  and  $g(x)$  in the table at right, to complete the table of values for their respective inverse functions,  $f^{-1}(x)$  and  $g^{-1}(x)$ . Then, use them to answer each of the questions below.

| $x$ | $f(x)$ | $g(x)$ |
|-----|--------|--------|
| 3   | 2      | -1     |
| 1   | -1     | 0      |
| -1  | 4      | 3      |

### To Find the Derivative of an Inverse Function:

Let  $f$  and  $g$  be inverse functions, such that  $f(g(x)) = x = g(f(x))$  where  $f(a) = b$  and  $g(b) = a$ .

To find  $g'(b)$  for a point  $(a, b)$  on  $f(x)$ .

- Find  $f'(x)$
- If you are only given  $b$ , set  $b = f(x)$  to find  $a$ .
- Find  $f'(a)$
- $g'(b) = \frac{1}{f'(a)}$

**NOTE:**  $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

\*\*\* Inverse functions have reciprocal slopes at corresponding points.\*\*\*

**EX #3:** Find  $g'(-1)$ ; where  $g(x)$  is the inverse of  $f(x)$ .

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|-----|--------|---------|--------|---------|
| 2   | 5      | 0       |        |         |
| 3   | -1     | -4      |        |         |
| -1  | 4      | 6       |        |         |

### The Existence of an Inverse Function:

- A function has an inverse function if and only if it is one-to-one.
- If  $f$  is strictly monotonic on its entire domain, (either increasing or decreasing on its entire domain) then it is one-to-one and therefore has an inverse function.

The relationship between a composite function and its inverse is stated below. If you use the chain rule and differentiate both sides, you can find a formula for the derivative of an inverse.

$$f[f^{-1}(x)] = x$$

There are questions where you will need to use a calculator to find approximations accurate to three decimal places. Here's an example of such a task.

**EX #5:** If  $f(x) = 4x^7 - x^2 + x - 5$ , find  $(f^{-1})'(3)$  accurate to three decimal places.



- Solve  $f(x) = 3$ , this will give you  $x = a$
- Differentiate  $f(x)$
- Evaluate  $f'(a)$  at the value from step 1, using all the decimal values, no early rounding.
- Use your result in the inverse derivative formula  $g'(b) = \frac{1}{f'(a)}$

### Derivatives of Other Bases

Let  $a$  be a positive real number ( $a \neq 1$ ) and let  $u$  be a differentiable function of  $x$ .

- $\frac{d}{dx}[a^x] = (\ln a)a^x$
- $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
- $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
- $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

**EX #7:** Find the derivative of each of the following.

A.  $y = \log_5 x$

B.  $y = \log_2(\sin x)$

C.  $y = \log(x + 3^x)$

D.  $y = \log_2 \sqrt[3]{2x + 1}$

**EX #8:** Find an equation of the tangent line of  $f(x) = 6^x$  where  $x = 2$ .

**EX #10:** Inverse Derivatives by Tables

Use the table at right to find each of the indicated values below. State the ordered pairs for the inverse function  $g^{-1}(x)$  in the column provided.

| $x$ | $f$ | $g$ | $f'$ | $g'$ | $g^{-1}$ |
|-----|-----|-----|------|------|----------|
| -3  | 2   | 5   | 0    | 4    |          |
| 0   | 5   | -1  | -5   | 1    |          |
| 2   | 4   | -3  | 1    | 2    |          |
| 4   | 2   | 2   | -1   | -3   |          |

A. Find  $[f^{-1}(4)]'$

B. Find  $[g^{-1}(-3)]'$

C. Write the equation for the line tangent to the graph of  $g^{-1}(x)$  when  $x = 2$ .

# NOTES

## 3.4

Inverse Functions  
Homework

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Problems 1 - 4, find  $(f^{-1})'(b)$  for each function.

1.  $f(x) = x^3 - 1, b = 7$

2.  $f(x) = x^3 + 2x - 1, b = 2$

3.  $f(x) = \sin x, b = \frac{1}{2}$  on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$

4.  $f(x) = \cos 2x, b = 1$  on  $[0, \frac{\pi}{2}]$

Problems 5 - 6, Multiple Choice.

5. If  $f(x) = \frac{4}{x} + 2$  and  $g$  is the inverse of  $f$ , then  $g'(10) =$ 

- A.
- $-16$
- B.
- $-\frac{1}{2}$
- C.
- $-\frac{1}{16}$
- D.
- $\frac{1}{2}$

6. If  $(3,5)$  is a point on the graph of  $y = f(x)$ , a one-to-one function, and  $f'(3) = 2$ , then which of the following must be true?

- A.  $\frac{d}{dx}[f^{-1}(3)] = \frac{1}{2}$       B.  $\frac{d}{dx}[f^{-1}(3)] = -\frac{1}{2}$
- C.  $\frac{d}{dx}[f^{-1}(5)] = \frac{1}{2}$       D.  $\frac{d}{dx}[f^{-1}(3)] = -2$

Problems 7 - 10, Short Answer.

7. If  $f(3) = 8$  and  $f'(3) = 5$  what do you know about  $f^{-1}$ ?8. Let  $f(x) = x^3 + x - 2$ , and let  $g$  be the inverse function. Evaluate  $g'(0)$ .9. A function  $f$  and its derivative take on the values shown in the table. If  $g$  is the inverse of  $f$ , find  $g'(6)$ .

| $x$ | $f(x)$ | $f'(x)$       |
|-----|--------|---------------|
| 2   | 6      | $\frac{1}{3}$ |
| 6   | 8      | $\frac{3}{2}$ |

10. Your parents bought you a brand new car at graduation! Well, we can dream, right? Assume that the value of the car is given by the function,  $V(t) = 25,000e^{-0.1t}$ , where  $v$  is the value, in dollars, of the car and  $t$  is the time, in years, from the date of purchase.

- A. How much was the car worth at the dealership ( $t = 0$ )?
- B. How much did the car depreciate during its fourth year (from  $t = 3$  to  $t = 4$ )?
- C. When you graduate from college, assuming you are on the 4-year plan, what is the instantaneous rate of change (in dollars per year) of the value at  $t = 4$ ? Why is this different than your answer in part B?
- D. When will the value of the car reach \$12,500?

Problems 11- 26 , Find the derivative of the function. Remember to use logarithmic properties before differentiating.

11.  $f(x) = 5^x$

12.  $y = 4^{3x}$

13.  $y = 6^{2x-3}$

14.  $y = x(7^x)$

15.  $y = x(3^{-2x})$

16.  $f(x) = x^2 3^x$

17.  $f(t) = \frac{5^{2t}}{t}$

18.  $f(\theta) = 2^\theta \cos(\pi\theta)$

21.  $f(x) = \log_3(x^2 - 4x)$

22.  $g(x) = \log_4(x^2 + 1)^3$

23.  $y = \log_3 \sqrt{x^2 + 1}$

24.  $y = \log_2 \sqrt[3]{2x - 4}$

25.  $f(x) = \log_2 \frac{x^2}{x+1}$

26.  $g(x) = \log \frac{x^2-1}{x}$

# Practice Sets

### 3.4 Inverse Functions Form A

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Find the derivative.

1.  $f(x) = 5^{3x}$

2.  $g(x) = \log_2(\cos x)$

3.  $y = x(2^{-x})$

4. If  $f(2) = -5$  and  $f'(2) = -\frac{3}{4}$ ;  
find  $[f^{-1}]'(-5)$ .

### 3.4 Inverse Functions Form B

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Find the derivative.

1.  $f(x) = 2^{5x}$

2.  $g(x) = \log_3 \sin x$

3.  $y = x(3^{-2x})$

4. If  $f(1) = 6$  and  $f'(1) = 4$ ;  
find  $[f^{-1}]'(6)$ .

### 3.4 Inverse Functions Form C

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Find the derivative.

1.  $f(x) = 7^{2x}$

2.  $g(x) = \log_2 \sqrt{x-3}$

3.  $y = x(5^{-3x})$

4. If  $f(-2) = 4$  and  $f'(-2) = \frac{2}{5}$ ;  
find  $[f^{-1}]'(4)$ .

### 3.4 Inverse Functions Form D

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Find the derivative.

1.  $f(x) = 2^{3x}$

2.  $g(x) = \log_5 \sqrt{x-3}$

3.  $y = x(3^{-4x})$

4. If  $f(3) = 7$  and  $f'(3) = 5$ ;  
find  $[f^{-1}]'(7)$ .

### 3.4 Transcendental Rules Form A

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Find the derivative. Simplify completely.

1.  $f(x) = (x^3)(e^{-x})$

2.  $g(x) = \cos(e^x)$

3.  $y = \ln(5x)^2$

4.  $f(x) = 5^{3x}$

### 3.4 Transcendental Rules Form B

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Find the derivative. Simplify completely.

1.  $f(x) = (2x)(e^{2x})$

2.  $g(x) = 2e^{\cos x}$

3.  $y = \ln(x-3)^2$

4.  $f(x) = -4^{2x}$

### 3.4 Transcendental Rules Form C

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Find the derivative. Simplify completely.

1.  $f(x) = (3x^2)(e^{-x})$

2.  $g(x) = \sin(e^{x^2})$

3.  $y = \ln(x-1)^2$

4.  $f(x) = 6^{-2x}$

### 3.4 Transcendental Rules Form D

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Find the derivative. Simplify completely.

1.  $f(x) = (-3x^2)(e^x)$

2.  $g(x) = e^{\sin 3x}$

3.  $y = \ln(x+2)^3$

4.  $f(x) = -e^{x^2}$

# Daily Lesson Quiz