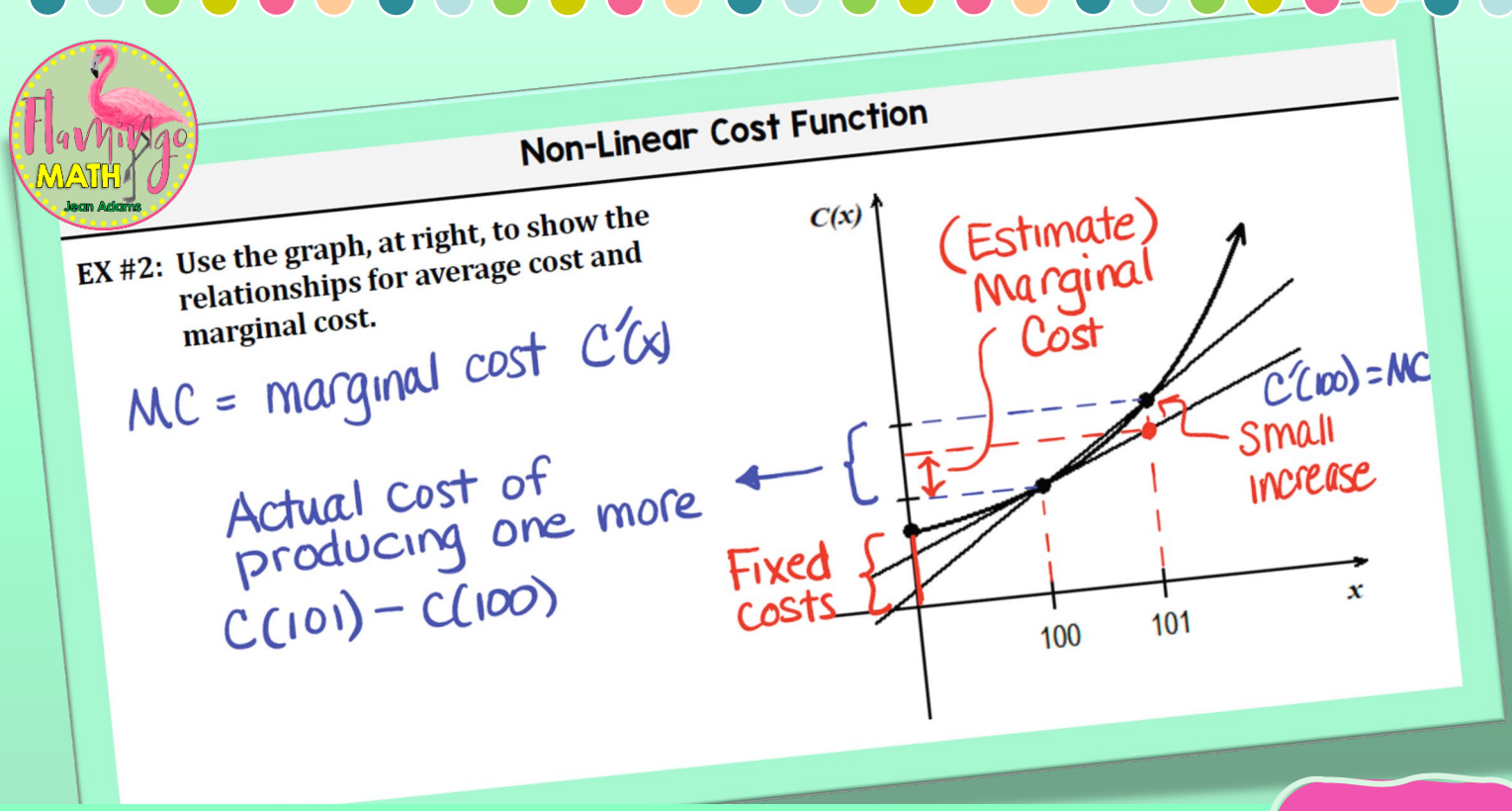
AP® Calculus

Contextual Applications of the Derivative



Guided Notes 4

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AP CALCULUS CURRICULUM

Scope and Sequence

Unit 4: Contextual Applications of Differentiation

Lesson	Skills & Learning Objectives
[4.1] Interpretations of the Derivative in Context	❑ Determine the meaning of a derivative as it relates to real-world scenarios
[4.2] Straight Line Motion: Connecting Position, Velocity & Acceleration	❑ Solve rectilinear motion problems involving speed, velocity and acceleration
[4.3] Rates of Change in Other Applications	❑ Apply the derivative to solve problems involving rates of change in applied contexts.
[4.4] Related Rates	❑ Calculate related rates in applied contexts ❑ Find a rate at which one quantity is changing by relating it to other quantities whose rates of change are known. ❑ Interpret related rates in applied contexts
[4.5] Local Linearity and Linearization	❑ Use tangent lines to approximate values near a point of tangency ❑ Determine whether a tangent line approximation overestimates or underestimates the corresponding function value
[4.6] L'Hospital's Rule	❑ Determine limits of functions that result in indeterminate forms.

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Lesson 2: Straight Line Motion
Connecting Position, Velocity & Acceleration

Topic 4.2: Straight Line Motion - Connecting Position, Velocity & Acceleration

In Unit 2, Lesson 7, the big ideas of position, velocity and acceleration were introduced. Particle motion problems deal with motion along a straight line. This includes horizontal and vertical motion of a falling object. It's important to note that this motion means the particle is moving along an axis, not along the graph of the function.

Vocabulary Review

The vocabulary terms for this lesson are restated here for your convenience.

Position	$s(t), x(t), y(t)$ tells where the particle is located at a given time, *t*.
Average Velocity	The average rate of change over $a \le t \le b$ $$v_{avg.} = \frac{\Delta s}{\Delta t} = \frac{\text{displacement}}{\text{elapsed time}} = \frac{s(b) - s(a)}{b - a} = m_{\text{secant}}$$
Instantaneous Velocity	$s'(t) = v(t)$ is rate of change of position at a given time and direction of movement. $$\frac{ds}{dt} = s'(t) = v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$
Speed	The rate at which the position is changing, |velocity| = $\|v(t)\| = \left\|\frac{ds}{dt}\right\|$
Acceleration	$s''(t) = v'(t) = a(t)$ tells how fast the velocity is changing at a given time. The sign tells if velocity is increasing or decreasing. $$s''(t) = \frac{d^2s}{dt^2} = v'(t) = \frac{dv}{dt} = a(t)$$
Displacement over an interval $[a, b]$	$s(b) - s(a)$ is the difference in distance from start to stop.
Distance Traveled	The sum of the absolute values of the distances between turning points.
Free Fall Motion	$s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$ is the motion of an object under the influence of gravity ignoring air resistance.
Direction changes	When velocity changes from either positive to negative or negative to positive at a point (time) . Velocity does not need to pass through zero or even be defined at a point where it's sign changes in order for direction to change.
Gravitational Constants	$9.8\ \frac{m}{sec^2}$ or $32\ \frac{ft}{sec^2}$

Table Question Stems

EX #2: The table below gives the velocity, in feet per second, of a particle moving along a horizontal axis. The velocity v is a twice-differentiable function of time t.

Time, t (sec)	0	2	3	5	7	10	12	15
Velocity, $v(t)$ (ft/sec)	-4	-1	1	2	3	5	3	6

A. At $t = 0$, determine whether the particle is moving to the right or to the left. Explain how you know.

B. Is there any time(s) during the interval $0 \leq t \leq 15$ seconds when the particle is at rest? Explain your answer.

C. Use the data from the table to find an approximation for $v'(8)$. Explain the meaning of $v'(8)$ in terms of the particle motion. Show the computations that lead to your answer, and indicate the units of measure.

D. Find the average acceleration of the particle for $10 \leq t \leq 12$ seconds. Explain what this number means in terms of the particle's velocity on the interval.

E. Let $a(t) = v'(t)$. Is there a time $t = c$ guaranteed such that $a(c) = 0$, in the interval $0 \leq t \leq 15$? Justify your answer.

EX #5: An office supply manufacturer produces spiral notebooks for interactive student notebooks. Find the quantity x that maximizes profit if the total revenue, $R(x)$, and the total cost $C(x)$, are given in dollars by

$$R(x) = 8x - 0.004x^2$$
$$C(x) = 700 + 1.2x$$

Where $0 \leq x \leq 1000$ notebooks. What production level gives the maximum profit?

EX #6: The table below shows cost and revenue for the production of a widget. Use the table to answer the following questions.

x	0	200	400	600	800	1000
$R(x)$	0	600	1200	1800	2400	3000
$C(x)$	800	1050	1200	1375	1450	1850

A. What is the price of the widget?

B. What are the fixed costs?

C. At what production level, x, is the profit for the widget maximized? Explain your reasoning.

EX #7: When a backpack production is 5000, marginal revenue is \$5 per backpack and marginal cost is \$4.65 per backpack. Would a maximum profit be expected to occur when production levels are above or below 5000 backpacks? Explain.

EX #6: A right circular cylinder is changing shape. The radius is decreasing at a rate of 2 inches/sec while its height is increasing at the rate of 5 inches/ sec. When the radius is 4 inches and the height is 6 inches, how fast is the volume changing?

Given: $V = \pi r^2 h$

1. KNOW:

2. GIVEN:

3. FIND:

EX #7: An inverted cone is leaking water at a rate of 1 cm^3/min. The cone has a height of 9 cm and a diameter of 6 cm. Find the rate at which the water level is dropping when $h = 3\ cm$.

Given: $V = \frac{1}{3}\pi r^2 h$

1. KNOW:

2. GIVEN:

3. FIND:

Lesson 5: Linear Approximation

Topic 4.6: Approximating Values of a Function Using Local Linearity and Linearization

In Unit 2 we briefly looked at the idea of local linearity, when zooming in on a function, we discovered as we zoom in on a function $y = f(x)$, the straight line that appears at a given point $x = a$ has a slope equal to the derivative $f'(a)$, and the equation for the tangent line at the given point was defined as:

$$f(x) = f(a) + f'(a)(x - a)$$

In this lesson, we will expand our investigation for a deeper understanding.

The Tangent Line Approximation

Suppose f is a differentiable function at a point $x = a$. Then, for values of x near a, the tangent line approximation to $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a)$$

We will define the expression $f(a) + f'(a)(x - a)$ as the ***local linearization of f near*** $\boldsymbol{x = a}$.
If you consider that point a is fixed, then $f(a)$ and $f'(a)$ are constant.
So , $f(x) \approx L(x) = f(a) + f'(a)(x - a)$ if x is close to a.
Graphically, linear approximation is $\Delta f \approx f'(a)\,\Delta x$, where $\Delta f = f(a + \Delta x) - f(a)$

The Linear Approximation Error

The **error**, $\boldsymbol{E(x)}$, in the linear approximation is simply the vertical "gap" between the actual curve and the tangent line. The error can be calculated by:

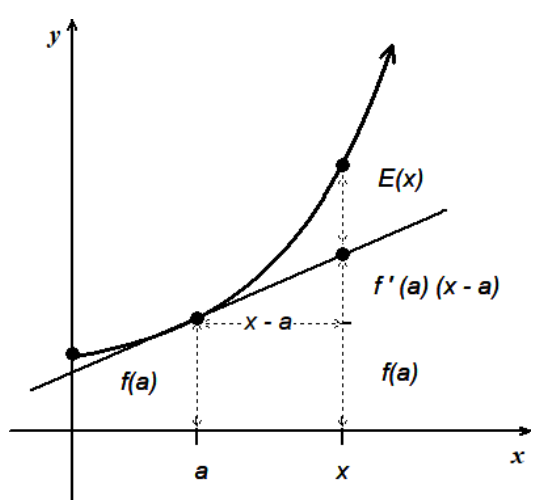
Error = |Exact − Approximation|

$$E(x) = |\Delta f - f'(a)\,\Delta x|$$

$$\text{Percentage Error} = \left|\frac{\text{exact} - \text{approximation}}{\text{exact}}\right| \times 100\%$$

Visual Vocabulary

EX #1: It can be shown that the best linear approximation to *f* near *a* is the tangent line approximation. Label the true value of $f(x)$, the tangent line , and the approximation.



Proper Notation and Protocol

In order to use L'Hospital's Rule on the AP Calculus Exam® students should show proper procedural steps and notation. To master the process be sure to . . .

1. Explicitly show the limit of the numerator is 0, ∞, or −∞.
2. Explicitly show the limit of the denominator is 0, ∞, or −∞.
3. You may want to state that the limit is indeterminate form because they are both 0 or ±∞.
4. By L'Hospital's Rule, write the limit as $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$
5. Find the limit using proper limit notation throughout.

Type I: Simple Polynomials and Powers

EX #1: $\lim_{x \to 5} \frac{2x^2 - 9x - 5}{x - 5}$

EX #2: $\lim_{x \to -5} \frac{\sqrt{4 - x} - 3}{x + 5}$

Lesson 7: Indeterminate Forms and L'Hospital's Rule

Topic 4.7: Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

Our use of indeterminate forms and applying L'Hospital's Rule in Calculus AB was limited to only $0/0$ and ∞/∞. When direct substitution left us with no clear answer, we began to analyze problems of this type. For instance:

Examples	Common thoughts about fractions that we have from Algebra:
$\lim\limits_{x\to 3} \dfrac{x-3}{x^2-9}$	If the numerator is zero, then the fraction is zero. If the denominator is zero, then the number does not exist.
$\lim\limits_{x\to \infty} \dfrac{3+2x-5x^3}{x^3-8}$	If the numerator is infinitely large, then the fraction is infinity. If the denominator is infinitely large, then the fraction is zero.

Problems arise as we take the limits for each situation. Some think that $0/0$ cancels out and the limit is 1, or we recall that infinity is not a real number, then calculus rules come to mind. That's why we define expressions of this type as "indeterminate forms." At first glance, it's not clear what happens to the limit, so we need further analysis. There are five more indeterminate forms that we will need to consider, just remember those algebraic rules can confuse your thinking in special cases that involve zero and infinity.

Algebra Rule:	$0(n)=0$	$n-n=0$	$n^0=1$		$n^\infty=\infty$
Will this work in Calculus?	$0(\pm\infty)$	$\infty-\infty$	0^0	∞^0	1^∞

EX #1: What do you remember about the first two examples? Use algebraic techniques to evaluate.

A. $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-9}$

B. $\lim\limits_{x\to \infty} \dfrac{3+2x-5x^3}{x^3-8}$