

AP[®] Calculus BC

Arc Length of a Plane Curve

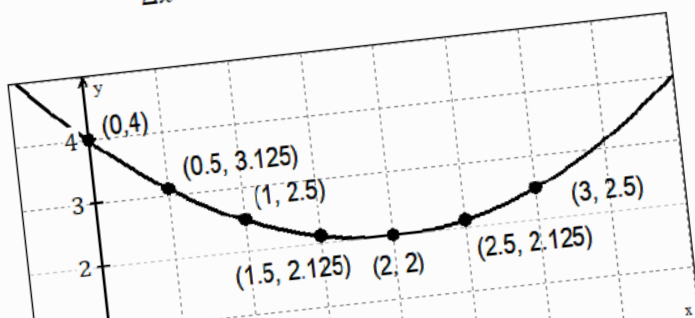


Lesson 8.8: Arc Length of a Plane Curve

APC 8.13: The Arc Length of a Smooth, Planar Curve and Distance Traveled (BC Only)

There are times when we need to find the length of a graph $y = f(x)$ on an interval $[a, b]$. In order to estimate the length of the curve, we can divide the interval up into sub-intervals and use the Pythagorean Theorem (or the distance formula) to find the lengths of each chord (hypotenuse) and sum them. Consider this “Big Idea” using the function below:

Exploration. Approximate the length of the curve $y = 0.5(x - 2)^2 + 2$ on the interval $[0, 3]$ where $\Delta x = 0.5$



Interval	Length

Completely Editable

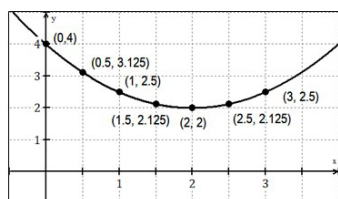
8.8

Lesson 8.8: Arc Length of a Plane Curve

Topic 8.13: The Arc Length of a Smooth, Planar Curve and Distance Traveled (BC Only)

There are times when we need to find the length of a graph $y = f(x)$ on an interval $[a, b]$. In order to estimate the length of the curve, we can divide the interval up into sub-intervals and use the Pythagorean Theorem (or the distance formula) to find the lengths of each chord (hypotenuse) and sum them. Consider this "Big Idea" using the function below:

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Interval	Length
Total	

Now, we are mathematicians and want to find the **EXACT** length of the curve! This process will lead us to making smaller intervals and taking the limit of the sums of lengths of line segments. Are you hearing a **definite integral** yet? Let's put a few restrictions on our functions, in order to guarantee that the integral will exist.

The Arc Length Problem

Suppose f is a **smooth function** on the interval $[a, b]$. Find the **arc length**, L , of the curve $y = f(x)$ over the interval $[a, b]$.

- ☐ **ARC LENGTH:** L on the interval $[a, b]$, where $(a, f(a))$ and $(b, f(b))$ are **endpoints**.
- ☐ **SMOOTH FUNCTION:** A function f is **smooth on an interval** if it has a derivative f' that is **continuous** throughout the interval.

Let's derive the formula to find lengths of a plane curve between two points:



Definition

If the function $y = f(x)$ is differentiable over the interval $a \leq x \leq b$, the **length of the curve** $y = f(x)$ from a to b is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

If the function $x = f(y)$ is differentiable over the interval $c \leq y \leq d$, the **length of the curve** $x = f(y)$ from c to d is

$$L = \int_c^d \sqrt{1 + (f'(y))^2} dy$$

EX #1: Find the length of the curve $f(x) = \frac{4\sqrt{x}}{3} x^{3/2} - 1$ over the interval $0 \leq x \leq 1$

EX #2: Find the length of the curve $y = \frac{2}{3}(x^2 + 1)^{3/2}$ from $x = 0$ to $x = 3$.

EX #3: Find the length of the curve $y = \frac{2}{3}x^{2/3}$ from $x = 1$ to $x = 8$. Round your answer to the nearest thousandth.



EX #4: Find the length of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ between $2 \leq y \leq 3$.

NOTES

8.8 Arc Length Homework

Name _____

Date _____ Period _____

Problems 1–2, Multiple Choice

1. Identify the definite integral that represents the arc length of the curve $y = \ln \sqrt{x}$ over the interval $[1, 3]$.

A. $L = \int_1^3 \sqrt{1 + \frac{1}{x^2}} dx$

B. $L = \int_1^3 \sqrt{1 + \frac{1}{4x^2}} dx$

C. $L = \int_1^3 \sqrt{1 + (\ln \sqrt{x})^2} dx$

D. $L = \int_1^3 \sqrt{1 + \frac{\ln x}{4x^2}} dx$

2. The length of a curve from $x = 1$ to $x = 5$ is given by $\int_1^5 \sqrt{1 + 9x^4} dx$. If the curve contains the point $(1, 4)$, which of the following could be an equation for this curve.

A. $y = \frac{1}{x^3} + 3$

B. $y = 3x^3 + 1$

C. $y = 5 - x^3$

D. $y = 6x + x^3$

Problems 3–10, Calculate the arc length over the given interval.

3. $y = 6 - 2x$, $[1, 3]$

4. $y = 2x^{3/2}$, $\left[\frac{1}{3}, \frac{5}{3}\right]$

5. $f(x) = 2(x - 1)^{3/2}$, $[1, 5]$

6. $y = \ln(\cos x)$, $\left[0, \frac{\pi}{4}\right]$

7. $y = \ln(\sin x)$, $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

8. $y = x^{2/3}$, $1 \leq y \leq 4$

9. $x = \frac{2}{3}(y - 1)^{3/2}$, $1 \leq y \leq 4$

10. $x = \frac{1}{3}(y^2 + 2)^{3/2}$, $0 \leq y \leq 1$

Problems 11–14, Find the length of the curves on the interval. Show the integral and evaluate with a calculator.

11. $(y - 1)^3 = x^2$, $(0, 1)$ to $(8, 5)$ and $y > 0$



12. $y = \frac{1}{2}(e^x + e^{-x})$, $[0, 2]$



Practice Sets

8.8 Arc Length of a Plane Curve Form A

Name _____

Date _____ Period _____

1. Write an expression involving the definite integral for the length of the curve given by $x = 1 + y^{3/2}$, $0 \leq y \leq 4$

2. Use the integral from question 1 to find the length of the arc.

8.8 Arc Length of a Plane Curve Form B

Name _____

Date _____ Period _____

1. Write an expression involving the definite integral for the length of the curve given by $y^2 = 4x^3$, $1 \leq x \leq 4$

2. Use the integral from question 1 to find the length of the arc.

8.8 Arc Length Form C

Name _____

Date _____ Period _____

1. Write an expression involving the definite integral for the length of the curve given by $x = 1 + 2y^{3/2}$, $0 \leq y \leq 4$

2. Use the integral from question 1 to find the length of the arc.

8.8 Arc Length Form D

Name _____

Date _____ Period _____

1. Write an expression involving the definite integral for the length of the curve given by $y^2 = 4x^3$, $0 \leq x \leq 2$.

2. Use the integral from question 1 to find the length of the arc.

Daily Lesson Quiz