

AP[®] Calculus

Analytical Applications of the Derivative

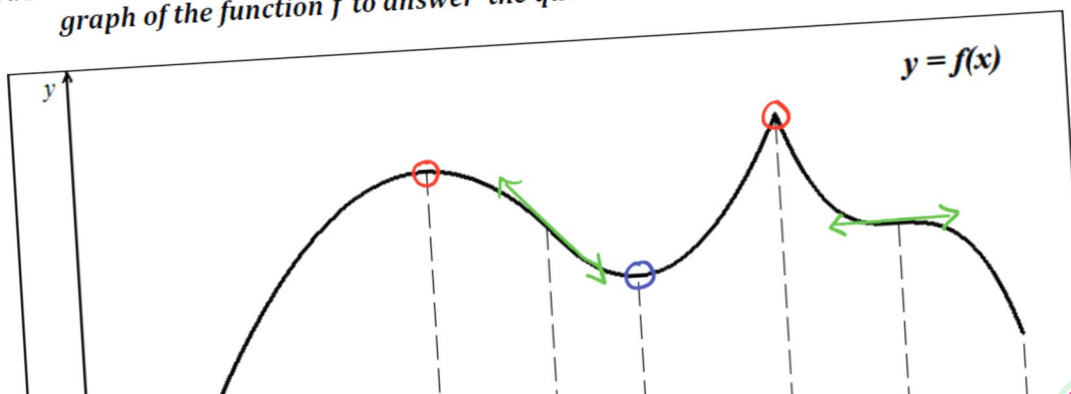


Lesson 3: The First Derivative Test Increasing and Decreasing Functions

Topic 5.3: Determining Intervals on Which a Function Is Increasing or Decreasing

Derivatives can be used to classify relative extrema as either relative minima or relative maxima. As we begin to focus on analyzing functions, we can determine some predictable and specific behaviors for functions in our calculus journey through “the mathematics of change.”

EX #1: This exploration leads to the conditions and outcomes of the first derivative test. Use the graph of the function f to answer the questions below.



Guided Notes

5

Completely Editable

AP CALCULUS CURRICULUM

Scope and Sequence

Unit 5: Analytic Applications of Differentiation

Lesson	Skills & Learning Objectives
[5.1] The Mean Value Theorem	☐ Apply the Mean Value Theorem to describe the behavior of a function over an interval
[5.2] Extrema on an Interval	☐ Use derivatives to find local and global extrema ☐ Use both open interval and closed interval methods for finding extrema. (Candidates Test) ☐ Use and apply the Extreme Value Theorem on intervals
[5.3] Increasing, Decreasing & First Derivative Test	☐ Use the first derivative test to find intervals of increase and decrease, maximum and minimum values
[5.4] Concavity & Second Derivative Test	☐ Use the second derivative test to find points of inflection, intervals where the function is concave up or concave down, and to find maximum and minimum values
[5.5] Sketching Graphs of Functions F , F' and F''	☐ Use the first and second derivative tests to analyze and sketch functions
[5.6] Optimization	☐ Optimize real-world situations by finding values that maximize or minimize constraints

Lesson 2: Extrema on an Interval

Topic 5.2: Extreme Value Theorem, Global Versus Local Extrema, and Critical Points

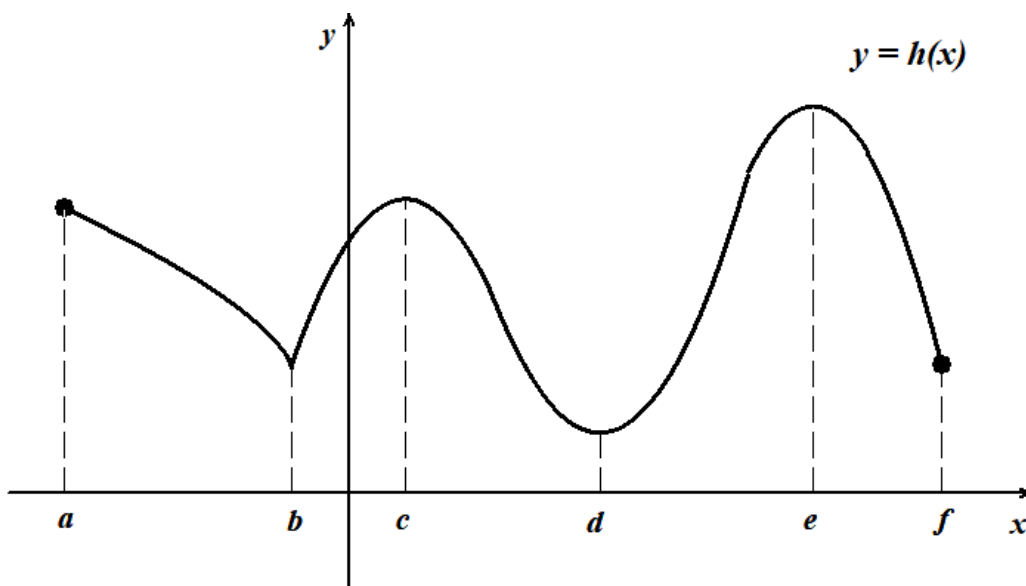
The maximum and minimum values (max and min) of a function are called the **extreme values** or **extrema** (singular: extremum) of a function. The process used to find them is referred to as **optimization**. There will be times when we want to find the max and min for x on the entire domain, and at other times on a particular interval. In this lesson, we will learn how to use calculus to find extrema, rather than a calculator.

Definition: (Absolute/Global) Extrema

If f is a function on an interval I , then $y = f(c)$ is the

- I. **(Absolute/Global) Maximum** on I , if and only if $f(c) \geq f(x)$ for all x in I .
- II. **(Absolute/Global) Minimum** on I , if and only if $f(c) \leq f(x)$ for all x in I .

EX #1: The graph of $y = h(x)$ is shown below. Determine the extrema of $h(x)$ on the interval $a \leq x \leq f$.



Defining Relative and Absolute Extrema of a Function

Relative or Local Extrema	y -values on the graph of a function where the function changes from increasing to decreasing and vice versa.
Absolute or Global Extrema	the highest or lowest y -values on the graph of a function or on a specified domain of a function.

Definition of Relative/Local Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum of f** .
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum of f** .

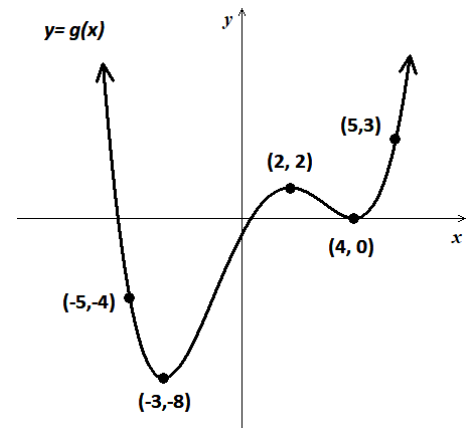
NOTEWORTHY THOUGHTS:

- ❑ When we say an “open interval containing c ” we mean that we can find some interval (a, b) , not including the endpoints, such that $a < c < b$.
- ❑ That is, c will be contained somewhere inside the interval and will not be either of the endpoints.
- ❑ A relative extrema is slightly different than an absolute extrema.
- ❑ All that’s necessary for a point to be a relative max or min is for that point to be a maximum or minimum in some interval of x ’s around $x = c$.
- ❑ There can be smaller or larger values of the function at some other location, but relative to $x = c$, $f(c)$ is smaller or larger than all the other function values in the “neighborhood of c .”

In many cases, we will be asked to analyze both open interval and closed interval functions.

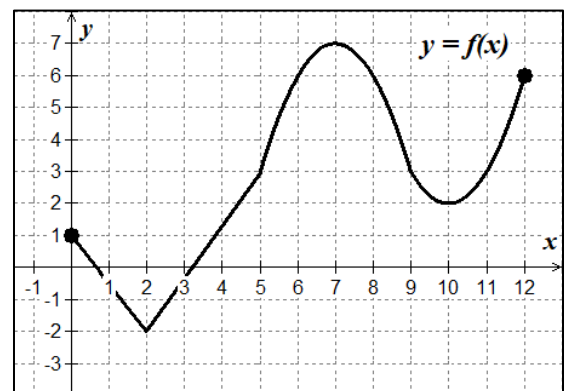
EX #2: Consider the graph of $y = g(x)$, shown at right.

- A. State the relative extrema of $g(x)$
- B. Considering the domain for g , what are the absolute extrema of the function?



EX #3: Consider the graph of $y = f(x)$ on $0 \leq x \leq 12$, shown below.

- A. Find the relative extrema of $f(x)$.
- B. Considering the domain for f , what are the absolute extrema of the function?



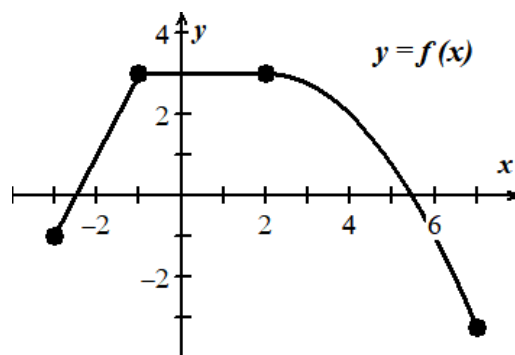
4. Using a **GREEN** colored pencil, sketch a tangent line, if possible, at the following points and determine their sign.
- A. At point b , $f'(b)$ is _____. Describe the change in the behavior of the graph of f at this value. _____
- B. At point d , $f'(d)$ is _____. Describe the change in the behavior of the graph of f at this value. _____
- C. At point e , $f'(e)$ is _____. Describe the change in the behavior of the graph of f at this value. _____
- D. Of these three points which two are points of inflection? _____
5. If a function f has a relative extremum at a point, then the derivative at the point is either zero or undefined. Is the converse of the statement also true? _____
 What point(s) justifies your answer? _____

Definitions: Increasing and Decreasing Functions

- ☐ A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- ☐ A function f is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

EX #2: The graph of $y = f(x)$, shown at right, is defined on $[-3, 7]$. List the *open intervals* over which the function is increasing, decreasing, or constant.

- A. $f(x)$ is increasing on _____
- B. $f(x)$ is decreasing on _____
- C. $f(x)$ is constant on _____



NOTE: A function is said to be **MONOTONIC** on an interval if its first derivative (which need not be continuous) does not change sign.

Topic 5.4: Using the First Derivative Test to Determine Relative (Local) Extrema

FIRST DERIVATIVE TEST:

Let c be a critical number of a function f that is continuous on an open interval. If f is differentiable on the interval, except possibly at c , then $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from negative to positive at c , then $f(c)$ is a **relative minimum** of f .
2. If $f'(x)$ changes from positive to negative at c , then $f(c)$ is a **relative maximum** of f .
3. If $f'(x)$ does not change signs at c , then $f(c)$ is neither a **relative minimum** nor a **relative maximum** of f .

We will explore how to find points of inflection and intervals where a function is concave up or concave down in the next lesson. The following chart will be helpful in summarizing and ***using correct vocabulary*** when justifying conditions:

Function Behavior				
$f(x)$				
$f'(x)$			$+$	$-$
$f''(x)$	$+$	$-$		

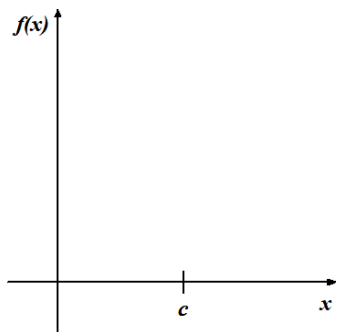
Symbol	Vocabulary
$+$	Positive
$-$	Negative
	Increasing
	Decreasing
	Concave Up
	Concave Down

EX #4: Given the function: $f(x) = x^4 - 8x^2 + 1$

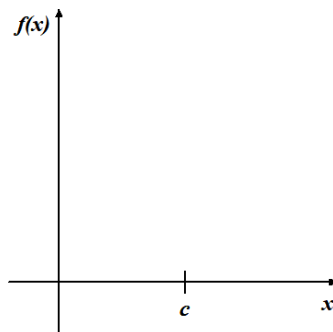
- Find the open intervals on which $f(x)$ is increasing and/or decreasing. Justify.
- Determine the x -values of any local extrema of $f(x)$. Justify.

EX #5: For each statement below, sketch a function that represents the results of the first derivative test for a continuous function $y = f(x)$ with a critical number at $x = c$.

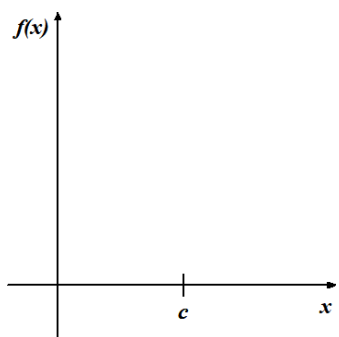
- A. $f(x)$ has an absolute minimum at $x = c$, because $f'(x)$ changes sign from negative to positive at $x = c$.



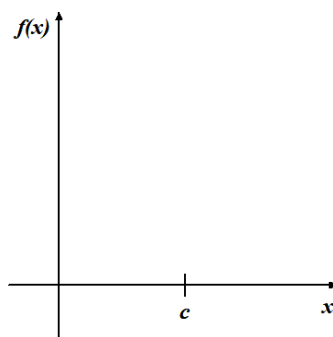
- B. $f(x)$ has a local maximum at $x = c$, because $f'(x)$ changes sign from positive to negative at $x = c$.



- C. $f(x)$ has no local extrema at $x = c$ because $f'(x) > 0$ for all $x > 0$ and is strictly monotonic.



- D. Since $f'(x)$ is negative immediately on either side of $x = c$, the function $f(x)$ has neither a local maximum nor local minimum at $x = c$.



EX #6: Find the exact values of any relative extrema of the function $g(x) = \frac{1}{2}x - \sin x$ on the interval $[0, 2\pi]$. Justify.

ANSWER THE QUESTION BEING ASKED . . . Are you looking for extrema (Y-VALUES) or the location where the extrema occurs (X-VALUES)?

Lesson 4: Concavity and the Second Derivative Test

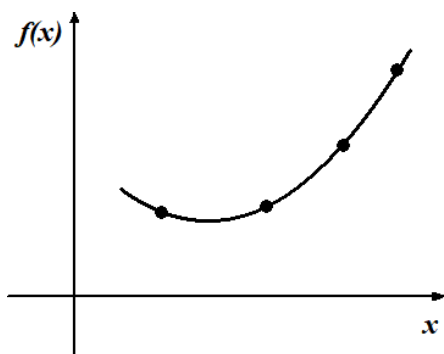
Topic 5.6: Determining Concavity of Functions over Their Domains

In our last lesson, we found intervals on which a function was increasing or decreasing and discovered how the first derivative describes the direction of a function (how the y -values are changing). In this lesson, we will see that the derivative of $f'(x)$, that is, the second derivative, will describe the concavity of the original function. Concavity tells us the direction of the curve, how it bends . . .

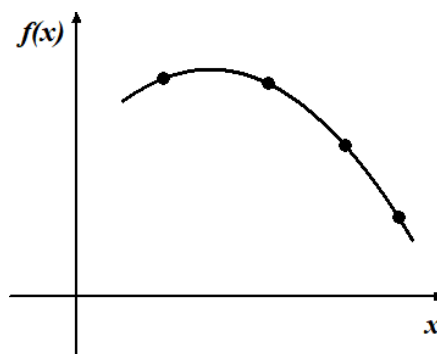
Just like direction, concavity of a curve can change, too. The points of change are called inflection points.

Concavity Exploration:

Draw small tangent lines at points along the curves below. What do you notice about the slopes of the tangent lines (the derivatives) as you move from left to right at these points?



A. Tangent lines lie below graph of f .



B. Tangent lines lie above graph of f .

Concavity Definition:

Let f be differentiable on an open interval I . The graph of f is:

- ☐ Concave upward on I if $f'(x)$ is increasing on the interval I ; and,
- ☐ Concave downward on I if $f'(x)$ is decreasing on the interval I .

Test for Concavity

- ☐ If $f''(x) > 0$, then graph of f is concave up.
- ☐ If $f''(x) < 0$, then graph of f is concave down.