

AP[®] Calculus BC

Alternating Series



Harmonic Series Exploration

1. Consider the idea that you wanted to have the signs alternate on each term. What would you need to do to the formula to cause this to happen?

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)(-1)^n$$

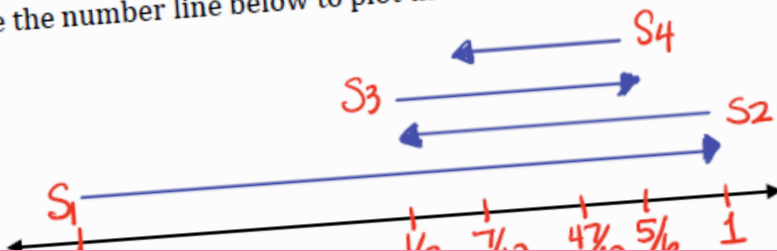
for $S_1 < 0$

or

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)(-1)^{n-1}$$

for $S_1 > 0$

2. Use the number line below to plot the sum of your new series.



$$\begin{aligned} * S_1 &= 1 \\ S_2 &= 1 - \frac{1}{2} = \frac{1}{2} \\ S_3 &= 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \\ S_4 &= \frac{5}{6} - \frac{1}{4} = \frac{7}{12} \\ S_5 &= \frac{7}{12} + \frac{1}{5} = \frac{47}{60} \end{aligned}$$

Sums oscillate back + forth

Completely Editable

10.4

Lesson 10.4: Alternating Series

Topic 10.5: Harmonic Series and p-Series

This section will introduce us to alternating series that contain both positive and negative terms. Let's start by revisiting the harmonic series.

Harmonic Series Exploration

Given the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

1. Consider the idea that you wanted to have the signs alternate on each term. What would you need to do to the formula to cause this to happen?

2. Use the number line below to plot the sum of your new series.



Topic 10.7: Alternating Series Test for Convergence

So, by definition, an alternating series is a series whose terms are alternately positive and negative. Alternating series can occur in two ways: either the odd terms are negative, or the even terms are negative.

Alternating Series Test

The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge if the following **conditions are met**:

1. Every a_n is positive
2. $\lim_{n \rightarrow \infty} a_n = 0$
3. $a_{n+1} < a_n$, for all n

Verbally Speaking:

A series converges if its terms:

1. Alternate in sign,
2. Decrease in magnitude; and
3. Have a limit of zero.

WORD OF CAUTION: This does NOT say that if $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges by the Alternating Series Test. The Alternating Series Test can **ONLY** be used to prove **convergence**. It does say that if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series **diverges** by the ***n*th Term Test for Divergence**, **NOT** by the Alternating Series Test.

EX #1: Determine whether the following series converge or diverge.

A. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

B. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1}$

C. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$

D. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$

Alternating Series Error Bound

For some convergent alternating series, the partial sum S_N is a useful method to approximate the sum S of the series. We call the error that is involved in using $S \approx S_N$ the remainder, where $R_N = S - S_N$.

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder $|R_N|$ involved in approximating the sum S by S_N is less than or equal to the first unused term. That is, $|S - S_N| = |R_N| \leq a_{n+1}$.

In other words, if the three conditions (alternating, decreasing in magnitude, and having a limit of 0) are met, you can approximate the sum of the series by using the n th partial sum S_N , and your error will be bounded by the absolute value of the first truncated term.

Topic 10.10: Alternating Series Error Bound

Alternating Series Estimation Theorem Exploration

EX #2: If a series is alternating and converges absolutely, then the sum of the series can be approximated by adding the values of the terms, with an error of the absolute value of the first unused term.

For example, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ can be approximated to an error of 0.01 by adding together

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \frac{1}{64} + \frac{1}{81} = 0.828$$

1. The next unused term is _____. Therefore, the error in your estimation is _____.
2. Use your calculator to find the sum of the first 200 terms. _____.
3. What is the error in your estimation from the 200 terms? _____.

EX #3: Given the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

- A. Approximate the sum, S , of the series by using its first four terms.

- B. Explain why the estimate found in (A) differs from the actual value by less than $\frac{1}{100}$.

Topic 10.9: Determining Absolute or Conditional Convergence

Absolute and Conditional Convergence

A series can be classified as one of the following:

- ☐ **Absolute Convergence:** The series $\sum a_n$ converges **absolutely** if $\sum |a_n|$ converges.
- ☐ **Absolute Convergence Implies Convergence:** If $\sum |a_n|$ converges, then $\sum a_n$ also converges.
- ☐ **Conditional Convergence:** An infinite series $\sum a_n$ converges **conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

EX #3: Determine which of the following are absolutely convergent, conditionally convergent, or divergent.

A. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

B. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

C. $\sum_{n=1}^{\infty} (-1)^n (0.01)^n$

D. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2}$

NOTES

10.4

Alternating Series
Homework

Name _____

Date _____ Period _____

Problems 1 – 6, Determine the convergence or divergence of the series.

1.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

2.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^n}$$

3.
$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2}$$

4.
$$\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n [\ln n]}$$

5.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[n]{n}}$$

6.
$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 4}$$

Problems 7- 9, Use your calculator and give an answer correct to three decimal places.

7. A. Approximate the sum of the series by using the first four terms $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$.



B. What is the error in your approximation?

8. A. Use the first five terms to approximate the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.



B. Why does the estimate differ from the actual value by less than 0.009?

C. Find the interval in which the sum S must lie.

9. A. Approximate the sum of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ so that the error will be less than 0.001.



B. How many terms are needed? How do you know that your approximation is guaranteed to be within 0.001 of the exact value? Justify your answer.

Problems 10 – 15, Determine whether the series converges conditionally, absolutely, or diverges.

10.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-3)^2 + 1}$$

11.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

12.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n}$$

13.
$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+1}{3n+1} \right)$$

Practice Sets

10.4 Alternating Series

Form A

Name _____

Date _____ Period _____

Determine if the series converges or diverges. Give a reason for your answer.

1.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$$

Determine whether the series converges conditionally or absolutely or diverges. Justify your answer.

2.
$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n^2}$$

10.4 Alternating Series

Form B

Name _____

Date _____ Period _____

Determine if the series converges or diverges. Give a reason for your answer.

1.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$$

Determine whether the series converges conditionally or absolutely or diverges. Justify your answer.

2.
$$\sum_{n=1}^{\infty} \frac{(-5)^n}{7^{n+1}}$$

Daily Lesson Quiz